Bulletin of Fukuoka University of Education, Vol.53, PartIII, 1 - 6 (2004)

A remark on birth-and-death chains in a random environment

Dedicated to Professor Hideki Kamimura on occasion of his 60th birthday

Toshio NAKATA *
Department of Information Education
Fukuoka University of Education

(Received August 22, 2003)

Abstract

In this note we study a question of whether birth-and-death chains under a special setting of random environment are transient of recurrent. We discuss some critical argument corresponding to Konsowa's results (Konsowa, *Stat. Prob. Lett.*, **56** (2002), 193–197).

Keywords: birth-and-death chains, random environment, transient and recurrent

1 Introduction

Consider the sequence of independent random variables $\{\alpha_n\}$ with $\alpha_0 = 1$ and

$$\alpha_n = \begin{cases} a_n, & \text{with probability } 1/2, \\ b_n, & \text{with probability } 1/2, \end{cases} \quad \text{for } n \geq 1,$$

where two sequences $\{a_n\}$ and $\{b_n\}$ satisfy $0 < a_n,\ b_n < 1$ for $n \geq 1$ and

$$\lim_{n\to\infty} a_n = a_*, \quad \lim_{n\to\infty} b_n = b_*, \quad 0 < a_*, b_* < 1. \tag{1}$$

For $\{\alpha_n\}$ we study a random walk $\{X_n\}$ on nonnegative integers, that is, $X_0 = 0$ and for $n \geq 1$

$$\begin{aligned} &P(X_n = i+1 | X_{n-1} = i, X_{n-2} = i_{n-2}, \cdots, X_1 = i_1, X_0 = 0, \text{environment}\{\alpha_n\}) \\ &= \alpha_i, \quad i \geq 0, \\ &P(X_n = i-1 | X_{n-1} = i, X_{n-2} = i_{n-2}, \cdots, X_1 = i_1, X_0 = 0, \text{environment}\{\alpha_n\}) \\ &= 1 - \alpha_i, \quad i > 0. \end{aligned}$$

We call the random walk $\{X_n\}$ a birth-and-death chain in the random environment $\{\alpha_n\}$. Indeed if $a_n = b_n$ for $n \geq 1$ then the environment is deterministic,

* This work was supported by Grant-in-Aid for Young Scientists (B) No. 14740077 from MEXT Japan.

so that the walk is well-known as the original birth-and-death chain [4, e.g. Example 1.3.4]. Moreover if $a_n = b_n = 1/2$ for $n \geq 1$ the walk is the simple random walk. In this article, we study whether $\{X_n\}$ is transient or recurrent.

If both parameters a_n and b_n are constant, then independent random variables $\{\alpha_n\}$ are identically distributed. So we can apply a general theory of random walks in a random environment [9]. Since $\{\alpha_n\}$ are not identically distributed, we need to investigate the relationship between the behavior of parameters and the recurrence of the chains.

On the other hand, [7] and [5] studied a simple random walk (SRW) on random **N**-trees. Since the growth of **N**-tree is random, we can regard it as a birth-and-death chain in a random environment. In fact, their random environment is

$$\alpha_n = \begin{cases} 1/2, & \text{with probability } 1 - q_n, \\ 2/3, & \text{with probability } q_n. \end{cases} \quad \text{for } n \geq 1.$$

They proved that if $\liminf nq_n > 1/\log 2$ (resp. $\limsup nq_n < 1/\log 2$) then SRW is transient a.s. (resp. recurrent a.s.) Moreover [6] gave examples for SRW of either type when $\lim q_n = 1/\log 2$. Note that our setting is $q_n = 1/2$ for any n instead of introducing parameters $\{a_n\}$ and $\{b_n\}$. In this article we discuss some critical argument corresponding to their results.

2 Birth-and-death chains in a random environment

To verify whether the birth-and-death chains are transient of recurrent, we investigate the convergence of $\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1-\alpha_k}{\alpha_k}$ (see [7]), that is,

$$\text{A birth-and-death chain is} \begin{cases} \text{transient a.s.} & \text{if } \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1-\alpha_k}{\alpha_k} < \infty \text{ a.s.,} \\ \text{recurrent a.s.} & \text{if } \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1-\alpha_k}{\alpha_k} = \infty \text{ a.s.} \end{cases} \tag{2}$$

For $i \geq 1$ we have

$$E\left[\log \frac{1-\alpha_i}{\alpha_i}\right] = \frac{1}{2} \log \frac{(1-a_i)(1-b_i)}{a_i b_i}, \text{Var}\left[\log \frac{1-\alpha_i}{\alpha_i}\right] = \frac{1}{4}\left(\log \frac{b_i(1-a_i)}{a_i(1-b_i)}\right)^2. \tag{3}$$

Setting $\sigma_i^2 = \text{Var}\left[\log \frac{1-\alpha_i}{\alpha_i}\right]$ and $s_n^2 = \sum_{j=1}^{n} \sigma_j^2$, we quote the following mild theorem in the case of $\lim_{n\to\infty} s_n^2 = \infty$.

THEOREM 2.1 **[2, p.200 Theorem 4.10]** *For independent random variables*

$\{X_n\}$, assume $\lim_{n\to\infty} s_n^2 = \infty$. Then for arbitrary $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n (X_i - E[X_i])}{(s_n^2)^{1/2+\varepsilon}} = 0 \quad a.s.$$

Using the theorem, we have the following proposition.

PROPOSITION 2.1 *Then birth-and-death chains in the random environment $\{\alpha_i\}$ are transient (resp. recurrent) a.s. if $a_* + b_* > 1$ (resp. $a_* + b_* < 1$).*

Proof. First, we assume $\lim_{n\to\infty} s_n^2 < \infty$. Then $\lim_{i\to\infty} \sigma_i^2 = \frac{1}{4}\left(\log \frac{b_*(1-a_*)}{a_*(1-b_*)}\right)^2 = 0$. Hence we have $a_* = b_*$. So if $a_* + b_* > 1$ then $a_* = b_* > 1/2$. Since the random walk $\{X_n'\}$ with transition probability

$$P(X_{n+1}' = i+1 | X_n' = i) = 1 - P(X_{n+1}' = i-1 | X_n' = i) = a_* > 1/2$$

is transient, so is $\{X_n\}$ by the usual technique of coupling (see [3]). Using the same argument, we can see that $\{X_n\}$ is recurrent if $a_* + b_* < 1$.

Second, we assume $\lim_{n\to\infty} s_n^2 = \infty$. Then by Theorem 2.1, for any $\delta > 0$ there exists N such that if $n > N$

$$-\delta\left(s_n^2\right)^{1/2+\varepsilon} < \sum_{i=1}^n (X_i - E[X_i]) < \delta\left(s_n^2\right)^{1/2+\varepsilon} \quad \text{a.s.}$$

for any $\varepsilon > 0$. Consequently we have

$$\exp\left\{-\delta\left(s_n^2\right)^{1/2+\varepsilon} + \sum_{i=1}^n E\left[X_i\right]\right\} < \prod_{i=1}^n \frac{1-\alpha_i}{\alpha_i} < \exp\left\{\delta\left(s_n^2\right)^{1/2+\varepsilon} + \sum_{i=1}^n E\left[X_i\right]\right\} \text{ a.s.} \tag{4}$$

Calling RHS (resp. LHS) of Eqn. (4) x_n (resp. y_n), we apply x_n (resp. y_n) to Raabe's test for the check of Eqn. (2). Now we see

$$\frac{x_{n+1}}{x_n} = \exp\left\{\delta\left\{\left(s_{n+1}^2\right)^{1/2+\varepsilon} - \left(s_n^2\right)^{1/2+\varepsilon}\right\} + E\left[\log\frac{1-\alpha_{n+1}}{\alpha_{n+1}}\right]\right\}.$$

By Eqn. (1), it turns out that σ_{n+1}^2 is bounded. So using Taylor expansion, we have

$$\left(s_{n+1}^2\right)^{1/2+\varepsilon} - \left(s_n^2\right)^{1/2+\varepsilon} = \left(s_n^2\right)^{1/2+\varepsilon}\left\{\left(1 + \frac{\sigma_{n+1}^2}{s_n^2}\right)^{1/2+\varepsilon} - 1\right\} \tag{5}$$

$$= \left(s_n^2\right)^{1/2+\varepsilon}\left\{\left(\frac{1}{2}+\varepsilon\right)\frac{\sigma_{n+1}^2}{s_n^2} + O\left(\left(\frac{\sigma_{n+1}^2}{s_n^2}\right)^2\right)\right\} \to 0 \quad \text{as } n \to \infty.$$

On the other hand, $\lim_{n\to\infty} E\left[\log \frac{1-\alpha_{n+1}}{\alpha_{n+1}}\right] = \frac{1}{2}\log\left(\frac{(1-a_*)(1-b_*)}{a_* b_*}\right) \neq 0$, because $a_* + b_* \neq 1$. Using Eqn. (3), we obtain

$$\lim_{n\to\infty} n\left(1 - \frac{x_{n+1}}{x_n}\right) = \lim_{n\to\infty} n\left[1 - \left(\frac{(1-a_n)(1-b_n)}{a_n b_n}\right)^{1/2}\right] \tag{6}$$

$$= \lim_{n\to\infty} n\left[1 - \left(1 + \frac{1-a_n-b_n}{a_n b_n}\right)^{1/2}\right] = \begin{cases} -\infty, & \text{if } a_* + b_* > 1, \\ +\infty, & \text{if } a_* + b_* < 1. \end{cases}$$

Similarly the same result holds for y_n. This completes the proof of the proposition. ∎

In the above proof, we use the same technique of [5]. However note that we do not need the sharp theorem of the law of iterated logarithm for proving it.

Next, we assume that $a_* + b_* = 1$. Setting

$$a_n = a_* + \xi_n, \quad b_n = b_* + \eta_n, \quad \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \eta_n = 0,$$

we study the relationship between ξ_n, η_n and the recurrence of the chains. Now we assume $\lim_{n\to\infty} s_n^2 < \infty$. Then using Kolmogorov's Convergence Criterion [8, e.g. p.212 Theorem 7.3.3], it turns out

$$\sum_{k=1}^{\infty}\left(\log\frac{1-\alpha_k}{\alpha_k} - E\left[\log\frac{1-\alpha_k}{\alpha_k}\right]\right) \text{ converges a.s.} \tag{7}$$

THEOREM 2.2 *Assume that* $a_* + b_* = 1$ *and* $\lim_{n\to\infty} s_n^2 < \infty$. *Then the birth-and-death chain is transient (resp. recurrent) a.s. if* $\limsup_{n\to\infty} n(\xi_n + \eta_n) < \frac{1}{2}$ *(resp.* $\liminf_{n\to\infty} n(\xi_n + \eta_n) > \frac{1}{2}$*).*

Since $\lim_{n\to\infty} s_n^2 < \infty$ holds under the condition $a_* + b_* = 1$, we deduce $a_* = b_* = 1/2$ using the same argument of the proof of Proposition 2.1.

Proof. Using Eqn. (7), there exist C_1 and C_2 such that

$$-\infty < C_1 < \sum_{i=1}^{n}\left(\log\frac{1-\alpha_i}{\alpha_i} - E\left[\log\frac{1-\alpha_i}{\alpha_i}\right]\right) < C_2 < \infty \quad \text{a.s.} \tag{8}$$

Hence

$$\exp\left\{C_1 + \sum_{i=1}^{n} E\left[\log\frac{1-\alpha_i}{\alpha_i}\right]\right\} < \prod_{i=1}^{n}\frac{1-\alpha_i}{\alpha_i} < \exp\left\{C_2 + \sum_{i=1}^{n} E\left[\log\frac{1-\alpha_i}{\alpha_i}\right]\right\} \quad \text{a.s.}$$

We use Raabe's test in the same way as the proof of Proposition 2.1. Assume that the limit of $n\left(1-x'_{n+1}/x'_n\right)$ exists, where $x'_n = \exp\left\{\sum_{i=1}^n E\left[\log\frac{1-\alpha_i}{\alpha_i}\right]\right\}$. Then

$$\begin{aligned}
&\lim_{n\to\infty} n\left(1-\frac{x'_{n+1}}{x'_n}\right) = \lim_{n\to\infty} n\left\{1-\exp E\left[\log\frac{1-\alpha_{n+1}}{\alpha_{n+1}}\right]\right\} \\
=\ & \lim_{n\to\infty} -\frac{n}{2}\log\frac{(1-a_n)(1-b_n)}{a_n b_n} = \lim_{n\to\infty}\frac{n}{2}\log\frac{(1/2+\xi_n)(1/2+\eta_n)}{(1/2-\xi_n)(1/2-\eta_n)} \\
=\ & \lim_{n\to\infty}\frac{n}{2}\log\left(1+\frac{\xi_n+\eta_n}{(1/2-\xi_n)(1/2-\eta_n)}\right) = \lim_{n\to\infty}\frac{n}{2}\log\left(1+\frac{1}{\frac{(1-2\xi_n)(1-2\eta_n)}{4(\xi_n+\eta_n)}}\right).
\end{aligned}$$

Note that $1-e^x = -x+O(x^2)$ for $|x| \ll 1$ for the second equality. Therefore if $\limsup_{n\to\infty} n(\xi_n+\eta_n) < \frac{1}{2}$, we see $\lim_{n\to\infty} n\left(1-\frac{x'_{n+1}}{x'_n}\right) < 1$. Hence Eqn. (4) is divergent, so that the chain is recurrent. On the other hand, if $\liminf_{n\to\infty} n(\xi_n+\eta_n) > \frac{1}{2}$, then we can see that the chain is transient. ∎

In the case of $\lim_{n\to\infty} s_n^2 = \infty$ and $a_* = b_* = 1/2$, we have the same result of Theorem 2.2, because we see that Eqn. (5) goes to 0 faster than $E[\log\{(1-\alpha_{n+1})/\alpha_{n+1}\}]$. However it is not tractable to classify the case of $\lim_{n\to\infty} s_n^2 = \infty$ and $a_* \neq b_* = 1-a_*$.

Under the condition of Theorem 2.2, if $\lim_{n\to\infty} n(\xi_n+\eta_n) = 1/2$, we have an example of either type corresponding to [6].

Example 2.1 If $\xi_n = \eta_n = 1/(4n)$ then $s_n^2 = 0$ and the birth-and-death chain is recurrent a.s. In fact, we only check the divergence of $\sum_n x'_n$ in the proof of Theorem 2.2. Since $x'_n = \prod_{i=1}^n (2i-1)/(2i+1) = 1/(2n+1)$, Eqn. (2) is divergent.

Example 2.2 If $\xi_n = \eta_n = (1+3/\log n)/(4n)$ then $s_n^2 = 0$ and the birth-and-death chain is transient a.s. Now we check $\sum_n x'_n < \infty$. Setting $v_n = \{n(\log n)^2\}^{-1}$, we see $\sum_n v_n < \infty$. Since $x'_n = \prod_{i=1}^n (2i-1-3/\log i)/(2i+1+3/\log i)$, we have

$$\frac{x'_n}{x'_{n+1}} = \frac{2n+3+3/\log(n+1)}{2n+1-3/\log(n+1)} = 1+\frac{1}{n}+\frac{3}{n\log n}+o\left(\frac{1}{n\log n}\right).$$

On the other hand,

$$\frac{v_n}{v_{n+1}} = \frac{(n+1)\{\log(n+1)\}^2}{n(\log n)^2} = \left(1+\frac{1}{n}\right)(1+\ell_n)^2 = 1+\frac{1}{n}+2\ell_n+o(\ell_n),$$

where $\ell_n = (\log(n+1)-\log n)/\log n < 1/(n\log n)$. So we obtain $x'_n/x'_{n+1} \geq v_n/v_{n+1}$ for sufficient large n. Therefore using the ratio comparison test [1, e.g. p.231 Theorem 6], $\sum_n x'_n < \infty$ holds. So is Eqn. (2).

References

[1] R. Buck, *Advanced Calculus*, Third Edition, McGraw-Hill (1978).

[2] K. Ito, *Probability (In Japanese)*, Iwanami 1991 (1991).

[3] Lindvall, *Lectures on the Coupling Method*, Wiley, (1992).

[4] J. Norris, *Markov Chains*, Cambridge UP (1997).

[5] M.H. Konsowa, Random walks on trees and the law of iterated logarithm, *Stat. Prob. Lett.* **56**, 193–197 (2002).

[6] M.H. Konsowa, An answer to an open question on random walks on trees, *Arabian J. Sci. Eng.* 451–455 (1996).

[7] M.H. Konsowa and J. Mitro, The type problem for random walks on trees, *J. Theoret. Prob.* **4** No.3, 535–550 (1991).

[8] S. Resnick, *A probability Path*, Birkhäuser (1999).

[9] F. Solomon, Random walks in a random environment, *Ann. Prob.*, **3** No.1, 1-31 (1975).