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On an $\{f, m; t, s\}$-max·hyper and a $\{k, m; t, s\}$-min·hyper in a Finite Projective Geometry $PG(t, s)$

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1. Introduction

We denote a finite projective geometry of t dimensions over the Galois field $GF(s)$ with s elements by $PG(t, s)$ where $t \geqq 2$ and s is a prime or prime power. The concept of an $\{f, m; t, s\}$-max·hyper (or a $\{k, m; t, s\}$-min·hyper) with weight $W=(w_1, w_2, \cdots, w_r)$ in $PG(t, s)$ has been introduced by N. Hamada and F. Tamari in [2]. It is known in [2, 3] that the concept of an $\{f, m; t, s\}$-max·hyper (or a $\{k, m; t, s\}$-min·hyper) is useful to investigate optimal linear codes over the Galois field and maximal t linearly independent sets.

In the present paper we get an upper bound on f for an $\{f, m; t, s\}$-max·hyper with weight $(w_1, w_2, \cdots, w_r)$ for given integers m, t and s, and show that there exist $\{f, m; t, s\}$-max·hypers which attain the upper bound for several given integers m, t and s.

Finally, we obtain a lower bound on k for a $\{k, m; t, s\}$-min·hyper for some given integers m, t and s.

2. An upper bound on f for an $\{f, m; t, s\}$-max·hyper for given integers m, t and s

Let F be a set of points $P_1, P_2, \cdots, P_k$ in $PG(t, s)$ and let $W=(w_1, w_2, \cdots, w_k)$ be an ordered set of positive integers $w_1, w_2, \cdots, w_k$. Let $(s^i-1)/(s-1)=v_i$ for any positive integer i. Let H_i $(i=1, 2, \cdots, v_{t+1})$ be a hyperplane in $PG(t, s)$ and let

(2.1) $$F \cap H_i=\{P_{l_{ij}}: j=1, 2, \cdots, \pi_i\}$$

for $i=1, 2, \cdots, v_{t+1}$.

DEFINITION 2.1. *(F, W) is said to be an $\{f, m; t, s\}$-max·hyper (or an $\{f, m; t, s\}$-min·hyper) with weight W if*

(2.2) $$\max_i \sum_{j=1}^{\pi_i} w_{l_{ij}}=m \quad (or \min_i \sum_{j=1}^{\pi_i} w_{l_{ij}}=m)$$

where $f=\sum_{r=1}^{k} w_r$ *and* $\min_i \sum_j w_{l_{ij}}=0$ *if* $F \cap H_i=\phi$ *for some* i.

Note that in the special case $t=2$, an $\{f, m; t, s\}$-max·hyper with weight $W=(1, 1,\cdots, 1)$ is an $\{f; m\}$-arc in $PG(2, s)$ (cf. [1, 2]).

Let $N=\|n_{ij}\|$ $(i=1, 2,\cdots, v_{t+1}, j=1, 2,\cdots, v_{t+1})$ be the incidence matrix of v_{t+1} hyperplanes H_i $(i=1, 2,\cdots, v_{t+1})$ and v_{t+1} points Q_j $(j=1, 2,\cdots, v_{t+1})$ in $PG(t, s)$, where

$$(2.3)\qquad n_{ij}=\begin{cases} 1, & \text{if the } i\text{th hyperplane } H_i \text{ contains the } j\text{th point } Q_j, \\ 0, & \text{otherwise.} \end{cases}$$

Let (F, W) be an $\{f, m; t, s\}$-max·hyper (or an $\{f, m; t, s\}$-min·hyper) with weight $W=(w_1, w_2,\cdots, w_r)$ where $F=\{Q_{i_k}\}$ $(k=1, 2,\cdots, r)$ and let

$$x_j=\begin{cases} w_k, & \text{if } j=i_k\ (k=1, 2,\cdots, r), \\ 0, & \text{otherwise} \end{cases}$$

for $j=1, 2,\cdots, v_{t+1}$. Then it is easy to see that

$$(2.4)\qquad \max_i \sum_{j=1}^{v_{t+1}} n_{ij}x_j=m \quad (\text{or } \min_i \sum_{j=1}^{v_{t+1}} n_{ij}x_j=m) \quad \text{and} \quad \sum_{j=1}^{v_{t+1}} x_j=f.$$

Conversely, let x_j $(1\leqq j\leqq v_{t+1})$ be nonnegative integers satisfying the condition (2.4) and let E be a set of integers j such as $x_j>0$, i.e., $E=\{i_1, i_2,\cdots, i_r\}$. Put $w_k=x_{i_k}$ $(k=1, 2,\cdots, r)$ and $F=\{Q_{i_k}\}$, then (F, W) is an $\{f, m; t, s\}$-max·hyper (or an $\{f, m; t, s\}$-min·hyper) with weight $W=(w_1, w_2,\cdots, w_r)$.

We prepare the following lemma in order to prove Theorem 2.2 and 2.4.

LEMMA 2.1. *Let (F_1, W_1) be an $\{f_1, m_1; t, s\}$-max·hyper with weight W_1 such that $f_1\geqq f$ for any $\{f, m_1; t, s\}$-max·hyper. If $1\leqq m_2<m_1$, then*

$$f_2\leqq f_1-(m_1-m_2)$$

for any $\{f_2, m_2; t, s\}$-max·hyper.

PROOF. Suppose that there exists an $\{f_2, m_2; t, s\}$-max·hyper (F_2, W_2) such that $f_2>f_1+m_2-m_1$ where $F_2=\{P_i\}$ $(i=1, 2,\cdots, k)$ and $W_2=(w_1, w_2,\cdots, w_k)$. Let H_i $(i=1, 2,\cdots, v_{t+1})$ be a hyperplane in $PG(t, s)$ and let $F_2\cap H_i=\{P_{l_{ij}}: j=1, 2,\cdots, \pi_i\}$ for $i=1, 2,\cdots, v_{t+1}$. Let α be an integer such that $\sum_{j=1}^{\pi_\alpha} w_{l_{\alpha j}}=m_2$ and put $w_i^*=w_i+m_1-m_2$ or w_i according as $i=l_{\alpha 1}$ or not. Put $W^*=(w_i^*)$. Then (F_2, W^*) is an $\{f^*, m_1; t, s\}$-max·hyper with weight W^* such that $f^*>f_1$ where $f^*=f_2+m_1-m_2$, which implies contradiction. This completes the proof.

Let

$$(2.5)\qquad m=t-1+\sigma_1v_1+\sigma_2v_2+\cdots+\sigma_{t-1}v_{t-1}+\sigma_tv_t$$

where σ_i's are integers such that $0\leqq\sigma_1\leqq s$, $0\leqq\sigma_i\leqq s-1$ $(i=2,\cdots, t-1)$ and $\sigma_t\geqq 0$. Then we obtain

THEOREM 2.2. *Let (F, W) be an $\{f, m; t, s\}$-max·hyper in $PG(t, s)$. Then*

(2.6) $$f \leqq t-1+\sigma_1 v_2+\cdots+\sigma_{t-1}v_t+\sigma_t v_{t+1}$$

where m is an integer given by (2.5).

PROOF. Let (F, W) be an $\{f, m; t, s\}$-max·hyper with weight $W=(w_1, w_2,\cdots, w_k)$ where $f=\sum_{r=1}^{k} w_r$, $m=t-1+\sum_{i=1}^{t}\sigma_i v_i$ and $F=\{P_1, P_2,\cdots, P_k\}$.
Let

$$F \cap H_i=\{P_{l_{ij}}: j=1,\cdots, \pi_i\} \quad (i=1, 2,\cdots, v_{t+1})$$

for an hyperplane H_i in $PG(t, s)$. Let $\mathscr{P}_t$ denote the set of all points in $PG(t, s)$. The proof is by induction on t.
(i) The case $t=2$. If $w_i \leqq \sigma_2$ for all $i=1, 2,\cdots, k$, then it follows from $k \leqq v_3$ that $f \leqq \sigma_2 v_3<1+\sigma_1 v_2+\sigma_2 v_3$. If there exists an integer β such that $w_\beta \geqq \sigma_2+1$, then consider the point P_β in F. Let $H_{\beta q}$ $(q=1, 2,\cdots, s+1)$ be $s+1$ lines in $PG(2, s)$ passing through the point P_β. Then $(H_{\beta q}-P_\beta)\cap(H_{\beta q'}-P_\beta)=\phi$ $(q \neq q')$ and $\bigcup_{q=1}^{s+1} H_{\beta q}=\mathscr{P}_2$. It follows from $\max_i \sum_j w_{l_{ij}}=1+\sigma_1 v_1+\sigma_2 v_2$ that

$$f=\sum_{r=1}^{k} w_r \leqq (1+\sigma_1 v_1+\sigma_2 v_2)(s+1)-s w_\beta$$
$$=1+\sigma_1 v_2+\sigma_2 v_3+(\sigma_2+1-w_\beta)s.$$

Since $w_\beta \geqq \sigma_2+1$, we have $f \leqq 1+\sigma_1 v_2+\sigma_2 v_3$. Hence Theorem 2.2 holds in this case.
(ii) The case $t=n+1$. Suppose that Theorem 2.2 is true in the case $t=n$. Then we shall prove Theorem 2.2 is true in the case $t=n+1$. Consider α such that

(2.7) $$\sum_{j=1}^{\pi_\alpha} w_{l_{\alpha j}}=n+\sigma_1 v_1+\cdots+\sigma_n v_n+\sigma_{n+1}v_{n+1}.$$

Let G_i $(i=1, 2,\cdots, v_{n+1})$ be an $(n-1)$-flat contained in H_α and let $F \cap G_i=\{P_{a_{ij}}: j=1, 2,\cdots, \lambda_i\}$. Suppose that $\max_i \sum_{j=1}^{\lambda_i} w_{a_{ij}} \leqq n-1+\sum_{j=1}^{n}\sigma_{j+1}v_j$, i.e., $\max_i \sum_{j=1}^{\lambda_i} w_{a_{ij}}=n-1+\sum_{j=1}^{n}\sigma_{j+1}v_j-\delta$ where δ is a nonnegative integer. Write $F^*=F\cap H_\alpha$ and $W^*=(w_{l_{\alpha j}})$. Since H_α can be identified with $PG(n, s)$, it follows that (F^*, W^*) is an $\{f^*, m^*; n, s\}$-max·hyper in $PG(n, s)$ where $f^*=\sum_{j=1}^{\pi_\alpha} w_{l_{\alpha j}}$ and $m^*=n-1+\sum_{i=1}^{n}\sigma_{i+1}v_i-\delta$. Therefore, it is easily shown from the assumption of induction and Lemma 2.1 that $\sum_{j=1}^{\pi_\alpha} w_{l_{\alpha j}} \leqq n-1+\sum_{i=1}^{n}\sigma_i v_{i+1}-\delta$. This contradicts (2.7).
Hence we have

$$\max_i \sum_{j=1}^{\lambda_i} w_{a_{ij}} \geqq n+\sum_{j=1}^{n}\sigma_{j+1}v_j.$$

Let β be an integer such that

(2.8) $$\sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}} \geqq n+\sum_{i=1}^{n}\sigma_{i+1}v_i.$$

Put $(F\cap H_{\gamma_i})-G_\beta=\{P_{d_{ij}}: j=1, 2,\cdots, \mu_i\}$ for $i=1, 2,\cdots, s+1$ where H_{γ_i}'s are hyperplanes in $PG(n+1, s)$ passing through G_β. Since $\{(F\cap H_{\gamma_i})-G_\beta\}\cap\{(F\cap H_{\gamma_j})-G_\beta\}=\phi$ $(i\neq j)$ and $\bigcup_{i=1}^{s+1}\{(F\cap H_{\gamma_i})-G_\beta\}=\mathscr{P}_{n+1}$, we obtain

$$\sum_{r=1}^{k} w_r=\sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}}+\sum_{i=1}^{s+1}\sum_{j=1}^{\mu_i} w_{d_{ij}} \tag{2.9}$$

and

$$\sum_{j=1}^{\pi_{\gamma_i}} w_{l_{\gamma_i j}}=\sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}}+\sum_{j=1}^{\mu_i} w_{d_{ij}}\leqq n+\sum_{j=1}^{n+1}\sigma_j v_j \tag{2.10}$$

for $i=1, 2,\cdots, s+1$.
Hence it follows from (2.8), (2.9) and (2.10) that

$$\begin{aligned} f=\sum_{r=1}^{k} w_r &\leqq \sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}}+\sum_{i=1}^{s+1}\{n+\sum_{j=1}^{n+1}\sigma_j v_j-\sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}}\} \\ &=(n+\sum_{j=1}^{n+1}\sigma_j v_j)(s+1)-s\sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}} \\ &\leqq n+\sum_{i=1}^{n+1}\sigma_i v_{i+1}\,. \end{aligned}$$

This completes the proof.

In the special case $t=2$, we have the following corollary:

COROLLARY 2.3. *In the case $t=2$ and $\sigma_2=0$,*

$$f\leqq 1+(m-1)(s+1) \tag{2.11}$$

for any $\{f, m; 2, s\}$-max·hyper in $PG(2, s)$.

REMARK. The bound (2.11) is well known for $\{f, m; 2, s\}$-max·hypers with weight $(1, 1,\cdots, 1)$, i.e., $\{f; m\}$-arcs. An $\{f; m\}$-arc which attains the bound (2.11) is said to be a maximal $\{f; m\}$-arc.

Let $M_1(t, s)$ be a set of integers m such that

$$m=t-1+\sigma_1 v_1+\sigma_2 v_2+\cdots+\sigma_{t-1}v_{t-1} \tag{2.12}$$

where σ_i's are integers such that $0\leqq\sigma_1\leqq s$, $0\leqq\sigma_i\leqq s-1$ $(i=2,\cdots, t-1)$ and $(\sigma_1, \sigma_2,\cdots, \sigma_{t-1})\neq(0, 0,\cdots, 0)$.

Let $M_2(t, s)$ be a set of integers m such that

$$m=t-1+\sigma_k^* v_k-\delta_k+\sum_{i=k+1}^{t-1}\sigma_i^* v_i \tag{2.13}$$

where k, δ_k and σ_k^*'s are integers such that $3\leqq k\leqq t-1$, $1\leqq\delta_k\leqq k-2$, $1\leqq\sigma_k^*\leqq s-1$ and $0\leqq\sigma_j^*\leqq s-1$ $(k+1\leqq j\leqq t-1)$.

It is easy to see that $M_1(t, s)\cap M_2(t, s)=\phi$ and $|M_1(t, s)|+|M_2(t, s)|=(s+1)s^{t-2}$

$-1+\sum_{k=3}^{t-1}(k-2)(s-1)s^{t-1-k}=v_t-(t-1)$ where $|A|$ denotes the cardinality of a set A. Hence an integer m ($t \leqq m \leqq v_t$) can be expressed uniquely as (2.12) or (2.13).

Let

(2.14) $$m^* = t-1+\sigma_k^* v_k - \delta_k + \sum_{i=k+1}^{t-1} \sigma_i^* v_i + \sigma_t^* v_t$$

where k, δ_k and σ_i^* ($k \leqq i \leqq t-1$) are integers given in (2.13) and σ_t^* is a nonnegative integer. Then we have

THEOREM 2.4. *Let (F, W) be an $\{f, m^*, t, s\}$-max-hyper in $PG(t, s)$ where m^* is an integer given by* (2.14). *Then*

(2.15) $$f \leqq t-1+\sigma_k^* v_{k+1} - \delta_k + \sum_{i=k+1}^{t-1} \sigma_i^* v_{i+1} + \sigma_t^* v_{t+1}\,.$$

PROOF. Let $m_1 = m^* + \delta_k$. It follows from Theorem 2.2 that

$$f_1 \leqq t-1+\sigma_k^* v_{k+1} + \sum_{i=k+1}^{t-1} \sigma_i^* v_{i+1} + \sigma_t^* v_{t+1}$$

for any $\{f_1, m_1; t, s\}$-max·hyper. Then we have from Lemma 2.1

$$f^* \leqq t-1+\sigma_k^* v_{k+1} - \delta_k + \sum_{i=k+1}^{t-1} \sigma_i^* v_{i+1} + \sigma_t^* v_{t+1}\,.$$

This completes the proof.

THEOREM 2.5. *Let σ be a nonnegative integer and let δ be an integer such that* $0 \leqq \delta \leqq t-1$. *Then*

(2.16) $$f \leqq \sigma v_{t+1} + \delta$$

for any $\{f, \sigma v_t + \delta;\ t, s\}$-max·hyper in $PG(t, s)$.

Proof is similar to that of Theorem 2.1 and hence omitted here.

3. A max·hyper which attain the upper bound (2.6), (2.15) or (2.16)

Let σ_i ($i=1, 2, \cdots, t-1$) be the integers given in (2.5) and let $\varepsilon_1 = s-\sigma_1$, $\varepsilon_i = s-1-\sigma_i$ ($2 \leqq i \leqq t-1$) and $D=\{\mu\colon \varepsilon_\mu \neq 0,\ 1 \leqq \mu \leqq t-1\}$. Let $\mathscr{B}$ be a set of ε_1 1-flats, ε_2 2-flats,$\cdots$, ε_{t-1} $(t-1)$-flats in $PG(t, s)$, i.e., let

$$\mathscr{B} = \{V_i^{(\mu)}\colon i=1, 2, \cdots, \varepsilon_\mu,\ \mu \in D\}$$

where $V_i^{(\mu)}$ ($i=1, 2, \cdots, \varepsilon_\mu$) denote (not necessarily distinct) ε_μ μ-flats in $PG(t, s)$ for each integer μ in D. In the special case $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{t-1}) = (0, 0, \cdots, 0)$, $\mathscr{B}$ is the empty set ϕ. Let $\eta_j(\mathscr{B})$ ($j=1, 2, \cdots, v_{t+1}$) be the number of flats $V_i^{(\mu)}$ ($1 \leqq i \leqq \varepsilon_\mu$, $\mu \in D$) in $\mathscr{B}$ which contain the point Q_j in $PG(t, s)$. Let us denote by $\mathscr{F}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{t-1};\ t, s)$, the family of all sets $\mathscr{B}$ which consist of ε_1 1-flats,$\cdots$, ε_{t-1} $(t-1)$-flats in $PG(t, s)$.

THEOREM 3.1. *Let m be an integer given by (2.5). If there is a set $\mathscr{B}$ in $\mathscr{F}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{t-1}; t, s)$ such that* $\max\{\eta_j(\mathscr{B})-1: 1 \leqq j \leqq v_{t+1}\} \leqq \sigma_t$, *then there exists a max·hyper which attains the upper bound (2.6).*

PROOF. As mentioned in Section 2, it is sufficient to show that there exists a set of nonnegative integers x_j $(1 \leqq j \leqq v_{t+1})$ satisfying the following conditions

$$\sum_{j=1}^{v_{t+1}} x_j = t-1+\sigma_1 v_2 + \cdots + \sigma_t v_{t+1}$$

and

$$\sum_{j=1}^{v_{t+1}} n_{ij} x_j \leqq t-1+\sigma_1 v_1 + \cdots + \sigma_t v_t$$

where n_{ij} is an integer given by (2.3). Let $\mathscr{B}$ be a set in $\mathscr{F}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{t-1}; t, s)$ such that $\max\{\eta_j(\mathscr{B})-1: 1 \leqq j \leqq v_{t+1}\} \leqq \sigma_t$. Put $x_j = \sigma_t + 1 - \eta_j(\mathscr{B})$ for $j = 1, 2, \cdots, v_{t+1}$. Since $\sum_{j=1}^{v_{t+1}} \eta_j(\mathscr{B}) = \sum_{i=1}^{t-1} \varepsilon_i v_{i+1}$, it follows that

$$
\begin{aligned}
(3.1) \qquad \sum_{j=1}^{v_{t+1}} x_j &= (\sigma_t+1)v_{t+1} - \sum_{j=1}^{v_{t+1}} \eta_j(\mathscr{B}) \\
&= (\sigma_t+1)v_{t+1} - \sum_{i=1}^{t-1} \varepsilon_i v_{i+1} \\
&= \sigma_t v_{t+1} + t - 1 + s v_2 + \sum_{i=2}^{t-1} (s-1) v_{i+1} - \sum_{i=1}^{t-1} \varepsilon_i v_{i+1} \\
&= t-1+(s-\varepsilon_1)v_2 + \sum_{i=2}^{t-1} (s-1-\varepsilon_i) v_{i+1} + \sigma_t v_{t+1} \\
&= t-1+\sigma_1 v_2 + \cdots + \sigma_t v_{t+1}.
\end{aligned}
$$

Let H_i and $V_j^{(\mu)}$ $(1 \leqq j \leqq \varepsilon_\mu, \mu \in D)$ be a hyperplane in $PG(t, s)$ and a μ-flat in $\mathscr{B}$, respectively. Since $|H_i \cap V_j^{(\mu)}| = v_\mu$ or $v_{\mu+1}$ for all i and j provided $\varepsilon_\mu \neq 0$, it follows that

$$(3.2) \qquad \sum_{j=1}^{v_{t+1}} n_{ij} \eta_j(\mathscr{B}) = \sum_{\mu \in D} \sum_{j=1}^{\varepsilon_\mu} |H_i \cap V_j^{(\mu)}| \geqq \sum_{\mu=1}^{t-1} \varepsilon_\mu v_\mu.$$

Thus from (3.2) we have

$$
\begin{aligned}
(3.3) \qquad \sum_{j=1}^{v_{t+1}} n_{ij} x_j &= (\sigma_t+1)v_t - \sum_{j=1}^{v_{t+1}} n_{ij} \eta_j(\mathscr{B}) \\
&\leqq t-1+(s-\varepsilon_1)v_1 + \cdots + (s-1-\varepsilon_{t-1})v_{t-1} + \sigma_t v_t \\
&= t-1+\sigma_1 v_1 + \cdots + \sigma_t v_t.
\end{aligned}
$$

Hence from (3.1) and (3.3) we have the required result.

THEOREM 3.2. *If there exists an $\{f, m; t, s\}$-max·hyper which attains the upper bound (2.6) where $m=t-1+sv_1+(s-1)v_2+\cdots+(s-1)v_{k-1}+(\sigma_k^*-1)v_k+\sum_{i=k+1}^{t}\sigma_i^*v_i$ and σ_i^* $(i=k, k+1,\cdots, t)$ is an integer given in (2.14), then there exists an $\{f^*, m^*; t, s\}$-max·hyper which attains the upper bound (2.15) where $m^*=t-1+\sigma_k^*v_k-\delta_k+\sum_{i=k+1}^{t}\sigma_i^*v_i$.*

PROOF. If there exists an $\{f, m; t, s\}$-max·hyper which attains the upper bound (2.6), there exists a set of nonnegative integers $\{x_j\}$ $(j=1, 2,\cdots, v_{t+1})$ which satisfy the conditions $\sum_{j=1}^{v_{t+1}} x_j=f$ and $\max_i \sum_{j=1}^{v_{t+1}} n_{ij}x_j=m$. Let α be an integer such that $\sum_{j=1}^{v_{t+1}} n_{\alpha j}x_j=m$. Consider an integer l such as $n_{\alpha l}=1$ and let $y_j=x_j+k-1-\delta_k$ or $y_j=x_j$ according as $j=l$ or not. Then we have

$$\begin{aligned}\sum_{j=1}^{v_{t+1}} y_j&=k-1-\sigma_k+\sum_{j=1}^{v_{t+1}} x_j\\&=t-1+sv_2+(s-1)v_3+\cdots+(s-1)v_k\\&\quad+k-1-\delta_k+(\sigma_k^*-1)v_{k+1}+\sum_{i=k+1}^{t}\sigma_i^*v_{i+1}\\&=t-1+\sigma_k^*v_{k+1}-\delta_k+\sum_{i=k+1}^{t}\sigma_i^*v_{i+1},\end{aligned}\tag{3.4}$$

since $v_{k+1}=k-1+sv_2+(s-1)v_3+\cdots+(s-1)v_k$.
We also have

$$\begin{aligned}\sum_{j=1}^{v_{t+1}} n_{ij}y_j&\leqq k-1-\delta_k+\sum_{j=1}^{v_{t+1}} n_{ij}x_j\\&\leqq t-1+sv_1+(s-1)v_2+\cdots+(s-1)v_{k-1}\\&\quad+k-1-\delta_k+(\sigma_k^*-1)v_k+\sum_{i=k+1}^{t}\sigma_i^*v_i\\&=t-1+\sigma_k^*v_k-\delta_k+\sum_{i=k+1}^{t}\sigma_i^*v_i\end{aligned}\tag{3.5}$$

for all $i=1, 2,\cdots, v_{t+1}$.
From (3.4) and (3.5), it follows that there exists a set of integers $\{y_j\}$ $(j=1, 2,\cdots, v_{t+1})$ such that $\sum_{j=1}^{v_{t+1}} y_j=t-1+\sigma_k^*v_{k+1}-\delta_k+\sum_{i=k+1}^{t}\sigma_i^*v_{i+1}$ and $\sum_{j=1}^{v_{t+1}} n_{ij}y_j\leqq t-1+\sigma_k^*v_k-\delta_k+\sum_{i=k+1}^{t}\sigma_i^*v_i$. Hence there exists an $\{f^*, m^*; t, s\}$-max·hyper which attains the upper bound (2.15) where $f^*=\sum_{j=1}^{v_{t+1}} y_j$ and $m^*=t-1+\sigma_k^*v_k-\delta_k+\sum_{i=k+1}^{t}\sigma_i^*v_i$. This completes the proof.

THEOREM 3.3. *Let σ be a nonnegative integer and let δ be an integer such that $0\leqq\delta\leqq t-1$. Then, there exists a $\{f, \sigma v_t+\delta; t, s\}$-max·hyper which attains the bound (2.16).*

PROOF. Let $x_1=\sigma+\delta$ and $x_j=\sigma$ for $j=2, 3,\cdots, v_{t+1}$. Then we obtain

$$\sum_{j=1}^{v_{t+1}} x_j=\sigma v_{t+1}+\delta \quad \text{and} \quad \sum_{j=1}^{v_{t+1}} n_{ij}x_j \leqq \sigma v_t+\delta \quad (1\leqq i\leqq v_{t+1}).$$

Hence we have the required result.

4. An upper bound on k for a $\{k, m; t, s\}$-min·hyper for given integers m, t and s

Let σ_i $(i=1, 2,\cdots, t)$ be any integers such that $0\leqq\sigma_1\leqq s$, $0\leqq\sigma_i\leqq s-1$ $(2\leqq i\leqq t-1)$ and $\sigma_t\geqq 0$ and let

(4.1) $$m=\sigma_1 v_1+\sigma_2 v_2+\cdots+\sigma_t v_t.$$

THEOREM 4.1. *Let m be an integer given by* (4.1) *and let* (K, W) *be a* $\{k, m; t, s\}$*-min·hyper in* $PG(t, s)$. *Then*

$$k\geqq\sigma_1 v_2+\cdots+\sigma_t v_{t+1}.$$

PROOF. Let (K, W) be a $\{k, m; t, s\}$-min·hyper with weight $W=(w_1, w_2,\cdots, w_q)$ in $PG(t, s)$ where $k=\sum_{r=1}^{q} w_r$ and $K=\{P_1, P_2,\cdots, P_q\}$. We shall prove this theorem by induction on t.

(i) The case $t=2$. Let P_γ be a point in K, then it follows from $\min_i \sum_j w_{l_{ij}}=\sigma_1+\sigma_2 v_2$ that $k\geqq(\sigma_1+\sigma_2 v_2)(s+1)-sw_\gamma=\sigma_1 v_2+\sigma_2 v_3+(\sigma_2-w_\gamma)s$ for $\gamma=1, 2,\cdots, k$. If there exists an integer γ such that $w_\gamma\leqq\sigma_2$, then $k\geqq\sigma_1 v_2+\sigma_2 v_3$. If $w_\gamma\geqq\sigma_2+1$ for all γ and $q=v_3$, then it follows from $0\leqq\sigma_1\leqq s$ that $k\geqq v_3(\sigma_2+1)>\sigma_1 v_2+\sigma_2 v_3$. If $w_\gamma\geqq\sigma_2+1$ for all γ and $q<v_3$, then there exists a point Q in $PG(t, s)$ which is not contained in K. Consider $s+1$ lines in $PG(2, s)$ passing through the point Q. Then we obtain

$$k\geqq(\sigma_1+\sigma_2 v_2)(s+1)\geqq\sigma_1 v_2+\sigma_2 v_3.$$

Hence Theorem 4.1 holds in this case.

(ii) The case $t=n+1$. Suppose Theorem 4.1 is true in the case $t=n$. We shall prove Theorem 4.1 is true in the case $t=n+1$. Let $F\cap H_i=\{P_{l_{ij}}: j=1, 2,\cdots, \pi_i\}$ $(i=1, 2,\cdots, v_{t+1})$ for a hyperplane H_i in $PG(t, s)$. Consider α such that $\sum_{j=1}^{\pi_\alpha} w_{l_{\alpha j}}=\sum_{i=1}^{n+1}\sigma_i v_i$. Let G_i $(i=1, 2,\cdots, v_{n+1})$ be an $(n-1)$-flat contained in H_α and let $F\cap G_i=\{P_{a_{ij}}: j=1, 2,\cdots, \lambda_i\}$. Using an argument similar to the proof of Theorem 2.2, we have

$$\min_i \sum_j w_{a_{ij}}\leqq\sum_{i=1}^{n}\sigma_{i+1}v_i.$$

Let β be an integer such that $\sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}}\leqq\sum_{i=1}^{n}\sigma_{i+1}v_i$. Using an argument similar to the proof of Theorem 2.2, we have

$$k=\sum_{r=1}^{q} w_r \geqq \sum_{j=1}^{\lambda_\beta} w_{a_{\beta j}}+\sum_{j=1}^{s+1}\{\sum_{i=1}^{n+1}\sigma_i v_i-\sum_{i=1}^{\lambda_\beta} w_{a_{\beta j}}\}$$
$$\geqq \sum_{i=1}^{n+2}\sigma_i v_{i+1}\,.$$

This completes the proof.

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