

On a New Multistep Method

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Abstract

In [1], [2], [3], [4], etc., Yanagiwara et al. derived the seven-node formulas, the six-node formulas, and the five-node formulas, for their multistep method, and they considered the stability of their formulas. But, in these papers their thinking was not exact concerning the minimum truncation error of their formulas. So, we want to consider a revised application of their formulas, and to show some examples.

1. Formulas

By the same method as the paper [1], we obtain the predictor formulas and the corrector formulas for our multistep method. For the case of the five-nodes and the seven-nodes, these formulas are given in [1]. But, we rewrite their formulas, and for the case of the six-nodes, we renew our formulas for the sake of the minimum truncation error. (In the case of the six-nodes at [1], the formula for y_{n-2} had the minimum truncation error, in the formulas for y_{n-3} , y_{n-2} , y_{n-1} , y_n , y_{n+1} , and y_{n+2} . So, we make the corrector formulas for y_{n-2} , y_{n-1} , y_n , y_{n+1} , y_{n+2} , and y_{n+3} . In this case, the formulas for y_{n+2} has the minimum truncation error.)

(1) five-node formulas

(a) correctors

$$y_{n-2} = y_n - \frac{h}{90} (29y'_{n-2} + 124y'_{n-1} + 24y'_n + 4y'_{n+1} - y'_{n+2}) - \frac{1}{90} h^6 y_n^{(6)} \quad (1.5.1)$$

$$y_{n-1} = y_n - \frac{h}{720} (-19y'_{n-2} + 346y'_{n-1} + 456y'_n - 74y'_{n+1} + 11y'_{n+2}) + \frac{11}{1440} h^6 y_n^{(6)} \quad (1.5.2)$$

$$y_{n+1} = y_n + \frac{h}{720} (11y'_{n-2} - 74y'_{n-1} + 456y'_n + 346y'_{n+1} - 19y'_{n+2}) + \frac{11}{1440} h^6 y_n^{(6)} \quad (1.5.3)$$

$$\begin{aligned}
y_{n+2} = y_n &+ \frac{h}{90} (-y'_{n-2} + 4y'_{n-1} + 24y'_n \\
&+ 124y'_{n+1} + 29y'_{n+2}) - \frac{1}{90} h^6 y_n^{(6)}
\end{aligned} \tag{1.5.4}$$

(b) predictor

$$\begin{aligned}
y_{n+3} = y_n &+ \frac{h}{80} (27y'_{n-2} - 138y'_{n-1} + 312y'_n \\
&- 198y'_{n+1} + 237y'_{n+2}) + \frac{51}{160} h^6 y_n^{(6)}
\end{aligned} \tag{1.5.5}$$

(2) six-node formulas

(a) correctors

$$\begin{aligned}
y_{n-2} = y_n &- \frac{h}{90} (28y'_{n-2} + 129y'_{n-1} + 14y'_n \\
&+ 14y'_{n+1} - 6y'_{n+2} + y'_{n+3}) + \frac{37}{3780} h^7 y_n^{(7)}
\end{aligned} \tag{1.6.1}$$

$$\begin{aligned}
y_{n-1} = y_n &- \frac{h}{1440} (-27y'_{n-2} + 637y'_{n-1} + 1022y'_n \\
&- 258y'_{n+1} + 77y'_{n+2} - 11y'_{n+3}) - \frac{271}{60480} h^7 y_n^{(7)}
\end{aligned} \tag{1.6.2}$$

$$\begin{aligned}
y_{n+1} = y_n &+ \frac{h}{1440} (11y'_{n-2} - 93y'_{n-1} + 802y'_n \\
&+ 802y'_{n+1} - 93y'_{n+2} + 11y'_{n+3}) - \frac{191}{60480} h^7 y_n^{(7)}
\end{aligned} \tag{1.6.3}$$

$$\begin{aligned}
y_{n+2} = y_n &+ \frac{h}{90} (-y'_{n-1} + 34y'_n \\
&+ 114y'_{n+1} + 34y'_{n+2} - y'_{n+3}) + \frac{1}{756} h^7 y_n^{(7)}
\end{aligned} \tag{1.6.4}$$

$$\begin{aligned}
y_{n+3} = y_n &+ \frac{h}{160} (3y'_{n-2} - 21y'_{n-1} + 114y'_n \\
&+ 114y'_{n+1} + 219y'_{n+2} + 51y'_{n+3}) - \frac{29}{2240} h^7 y_n^{(7)}
\end{aligned} \tag{1.6.5}$$

(b) predictor

$$\begin{aligned}
y_{n+4} = y_n &+ \frac{h}{45} (-14y'_{n-2} + 84y'_{n-1} - 196y'_n \\
&+ 344y'_{n+1} - 186y'_{n+2} + 148y'_{n+3}) + \frac{286}{945} h^7 y_n^{(7)}
\end{aligned} \tag{1.6.6}$$

(3) seven-node formulas

(a) correctors

$$y_{n-3} = y_n - \frac{h}{2240} (685y'_{n-3} + 3240y'_{n-2} + 1161y'_{n-1} + 2176y'_n - 729y'_{n+1} + 216y'_{n+2} - 29y'_{n+3}) - \frac{9}{896} h^8 y_n^{(8)} \quad (1.7.1)$$

$$y_{n-2} = y_n - \frac{h}{3780} (-37y'_{n-3} + 1398y'_{n-2} + 4863y'_{n-1} + 1328y'_n + 33y'_{n+1} - 30y'_{n+2} + 5y'_{n+3}) + \frac{1}{756} h^8 y_n^{(8)} \quad (1.7.2)$$

$$y_{n-1} = y_n - \frac{h}{60480} (271y'_{n-3} - 2760y'_{n-2} + 30819y'_{n-1} + 37504y'_n - 6771y'_{n+1} + 1608y'_{n+2} - 191y'_{n+3}) - \frac{191}{120960} h^8 y_n^{(8)} \quad (1.7.3)$$

$$y_{n+1} = y_n + \frac{h}{60480} (-191y'_{n-3} + 1608y'_{n-2} - 6771y'_{n-1} + 37504y'_n + 30819y'_{n+1} - 2760y'_{n+2} + 271y'_{n+3}) - \frac{191}{120960} h^8 y_n^{(8)} \quad (1.7.4)$$

$$y_{n+2} = y_n + \frac{h}{3780} (5y'_{n-3} - 30y'_{n-2} + 33y'_{n-1} + 1328y'_n + 4863y'_{n+1} + 1398y'_{n+2} - 37y'_{n+3}) + \frac{1}{756} h^8 y_n^{(8)} \quad (1.7.5)$$

$$y_{n+3} = y_n + \frac{h}{2240} (-29y'_{n-3} + 216y'_{n-2} - 729y'_{n-1} + 2176y'_n + 1161y'_{n+1} + 3240y'_{n+2} + 685y'_{n+3}) - \frac{9}{896} h^8 y_n^{(8)} \quad (1.7.6)$$

(b) predictor

$$y_{n+4} = y_n + \frac{h}{945} (286y'_{n-3} - 2010y'_{n-2} + 6054y'_{n-1} - 9836y'_n + 11514y'_{n+1} - 5622y'_{n+2} + 3394y'_{n+3}) + \frac{278}{945} h^8 y_n^{(8)} \quad (1.7.7)$$

(4) eight-node formulas

(a) correctors

$$y_{n-3} = y_n - \frac{h}{4480} (1325y'_{n-3} + 6795y'_{n-2} + 1377y'_{n-1} + 5927y'_n - 3033y'_{n+1} + 1377y'_{n+2} - 373y'_{n+3} + 45y'_{n+4}) + \frac{369}{44800} h^9 y_n^{(9)} \quad (1.8.1)$$

$$y_{n-2} = y_n - \frac{h}{3780} (-32y'_{n-3} + 1363y'_{n-2} + 4968y'_{n-1} + 1153y'_n + 208y'_{n+1} - 135y'_{n+2} + 40y'_{n+3} - 5y'_{n+4}) - \frac{127}{113400} h^9 y_n^{(9)} \quad (1.8.2)$$

$$y_{n-1} = y_n - \frac{h}{120960} (351y'_{n-3} - 4183y'_{n-2} + 57627y'_{n-1} + 81693y'_n - 20227y'_{n+1} + 7227y'_{n+2} - 1719y'_{n+3} + 191y'_{n+4}) + \frac{3233}{3628800} h^9 y_n^{(9)} \quad (1.8.3)$$

$$y_{n+1} = y_n + \frac{h}{120960} (-191y'_{n-3} + 1879y'_{n-2} - 9531y'_{n-1} + 68323y'_n + 68323y'_{n+1} - 9531y'_{n+2} + 1879y'_{n+3} - 191y'_{n+4}) + \frac{2497}{3628800} h^9 y_n^{(9)} \quad (1.8.4)$$

$$y_{n+2} = y_n + \frac{h}{3780} (5y'_{n-2} - 72y'_{n-1} + 1503y'_n + 4688y'_{n+1} + 1503y'_{n+2} - 72y'_{n+3} + 5y'_{n+4}) - \frac{23}{113400} h^9 y_n^{(9)} \quad (1.8.5)$$

$$y_{n+3} = y_n + \frac{h}{4480} (-13y'_{n-3} + 117y'_{n-2} - 513y'_{n-1} + 2777y'_n + 3897y'_{n+1} + 5535y'_{n+2} + 1685y'_{n+3} - 45y'_{n+4}) + \frac{81}{44800} h^9 y_n^{(9)} \quad (1.8.6)$$

$$y_{n+4} = y_n + \frac{h}{945} (8y'_{n-3} - 64y'_{n-2} + 216y'_{n-1} - 106y'_n + 1784y'_{n+1} + 216y'_{n+2} + 1448y'_{n+3} + 278y'_{n+4}) - \frac{107}{14175} h^9 y_n^{(9)} \quad (1.8.7)$$

(b) predictor

$$y_{n+5} = y_n + \frac{h}{24192} (-7155y'_{n-3} + 57515y'_{n-2} - 202815y'_{n-1} + 417735y'_n - 487225y'_{n+1} + 442305y'_{n+2} - 193365y'_{n+3} + 93965y'_{n+4}) - \frac{3240715}{3483648} h^9 y_n^{(9)} \quad (1.8.8)$$

(5) nine-node formulas

(a) correctors

$$\begin{aligned}
y_{n-4} = y_n - \frac{h}{14175} (4063y'_{n-4} + 22576y'_{n-3} + 244y'_{n-2} + 32752y'_{n-1} \\
- 9080y'_n + 9232y'_{n+1} - 3956y'_{n+2} + 976y'_{n+3} - 107y'_{n+4}) \\
- \frac{94}{14175} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.1}$$

$$\begin{aligned}
y_{n-3} = y_n - \frac{h}{44800} (-369y'_{n-4} + 16202y'_{n-3} + 57618y'_{n-2} \\
+ 34434y'_{n-1} + 33440y'_n - 9666y'_{n+1} + 3438y'_{n+2} \\
- 778y'_{n+3} + 81y'_{n+4}) + \frac{113}{89600} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.2}$$

$$\begin{aligned}
y_{n-2} = y_n - \frac{h}{113400} (127y'_{n-4} - 1976y'_{n-3} + 44446y'_{n-2} \\
+ 141928y'_{n-1} + 43480y'_n - 872y'_{n+1} - 494y'_{n+2} \\
+ 184y'_{n+3} - 23y'_{n+4}) - \frac{23}{113400} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.3}$$

$$\begin{aligned}
y_{n-1} = y_n - \frac{h}{3628800} (-3233y'_{n-4} + 36394y'_{n-3} - 216014y'_{n-2} \\
+ 1909858y'_{n-1} + 2224480y'_n - 425762y'_{n+1} + 126286y'_{n+2} \\
- 25706y'_{n+3} + 2497y'_{n+4}) + \frac{2497}{7257600} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.4}$$

$$\begin{aligned}
y_{n+1} = y_n + \frac{h}{3628800} (2497y'_{n-4} - 25706y'_{n-3} + 126286y'_{n-2} \\
- 425762y'_{n-1} + 2224480y'_n + 1909858y'_{n+1} - 216014y'_{n+2} \\
+ 36394y'_{n+3} - 3233y'_{n+4}) + \frac{2497}{7257600} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.5}$$

$$\begin{aligned}
y_{n+2} = y_n + \frac{h}{113400} (-23y'_{n-4} + 184y'_{n-3} - 494y'_{n-2} - 872y'_{n-1} \\
+ 43480y'_n + 141928y'_{n+1} + 44446y'_{n+2} - 1976y'_{n+3} \\
+ 127y'_{n+4}) - \frac{23}{113400} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.6}$$

$$\begin{aligned}
y_{n+3} = y_n + \frac{h}{44800} (81y'_{n-4} - 778y'_{n-3} + 3438y'_{n-2} - 9666y'_{n-1} \\
+ 33440y'_n + 34434y'_{n+1} + 57618y'_{n+2} + 16202y'_{n+3} \\
- 369y'_{n+4}) + \frac{113}{89600} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.7}$$

$$\begin{aligned}
y_{n+4} = y_n &+ \frac{h}{14175} (-107y'_{n-4} + 976y'_{n-3} - 3956y'_{n-2} \\
&+ 9232y'_{n-1} - 9080y'_n + 32752y'_{n+1} + 244y'_{n+2} \\
&+ 22576y'_{n+3} + 4063y'_{n+4}) - \frac{94}{14175} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.8}$$

(b) predictor

$$\begin{aligned}
y_{n+5} = y_n &+ \frac{h}{145152} (41705y'_{n-4} - 376570y'_{n-3} + 1512830y'_{n-2} \\
&- 3552370y'_{n-1} + 5425760y'_n - 5258830y'_{n+1} \\
&+ 3821570y'_{n+2} - 1493830y'_{n+3} + 605495y'_{n+4}) \\
&+ \frac{81385}{290304} h^{10} y_n^{(10)}
\end{aligned} \tag{1.9.9}$$

2. Our evaluation of truncation errors

In [1] Yanagiwara considered that the corrector for y_{n+1} has the minimum truncation error. This idea was not exact. He made a mistake by this idea.

In the cases which are more than five-nodes, the corrector for y_{n+2} has the minimum truncation error.

For the chief correctors of each of (1.5.3), (1.6.4), (1.7.5), (1.8.6), and (1.9.6), we must use the formula which has the minimum truncation error. In [1], solving the initial value problem, Yanagiwara decided the starting values:

for the stepsize: $h = 0.0625$ and

for the value at the judge of the convergence: $eps = 10^{-11}$.

These two values were derived to the loss of significant figures at the beginning steps.

Evaluating the minimum truncation error for $h = 2^{-3}$ and $h = 2^{-8}$ in the case from 5 nodes through 9 nodes concerning the numerical example I, we show the following **Figure 1** as an example. Here, the horizontal axis shows the value of x , and the vertical axis indicates the common logarithms of the absolute value of the truncation errors: $\log \left(\frac{11}{1440} h^6 |y^{(6)}| \right)$ at (1.5.3), etc.. The figures from 5 through 9 on or near the curves indicate the numbers of nodes. (Solving the initial value problem by the temporary values of the stepsize: h and the convergence judge: eps , we obtain the values of y at the calculating steps of x .)

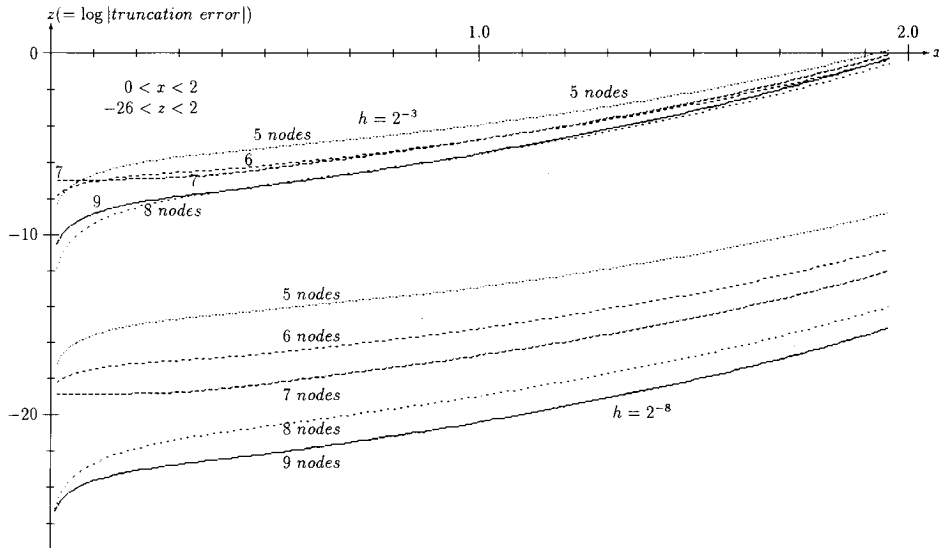


Figure 1

About the example II, evaluating the minimum truncation errors for $h = 2^{-2}$ and $h = 2^{-7}$ in the case from 5 nodes through 9 nodes, we have the following **Figure 2** as an example. Here, the horizontal axis gives the values of x , and the vertical axis points the common logarithms of the absolute value of the truncation errors: $\log \left(\frac{23}{113400} h^{10} |y^{(10)}| \right)$ in (1.9.6), etc..

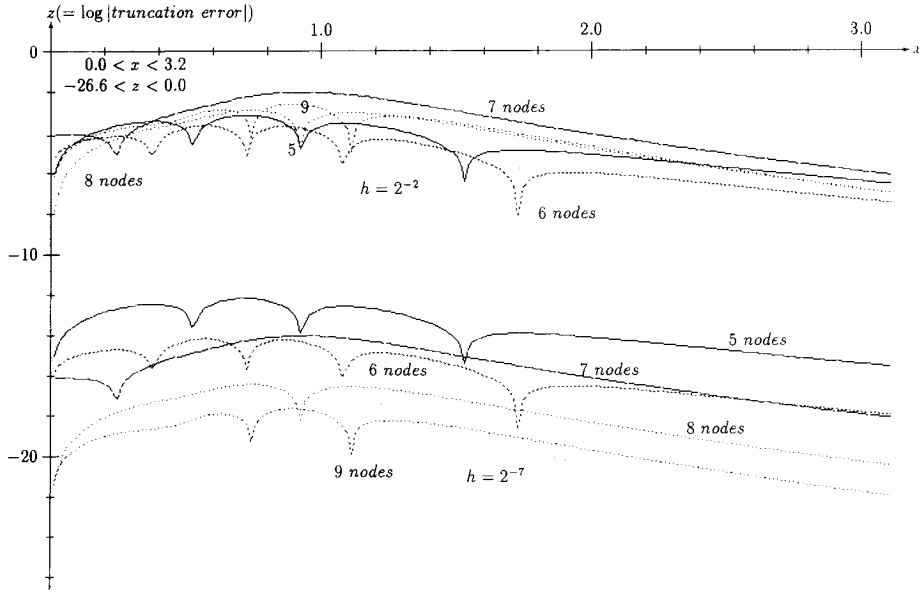


Figure 2

Using these above figures and other figures, we determine the stepsize: h and the convergence judge: eps , then we study the numerical example I and the numerical example II.

3. Numerical example I

We summarize the result in **Table 1-a**, **Table 1-b** (by (1.6.1) to (1.6.6)), which are at the end of this paper, **Figure 3** (by (1.7.1) to (1.7.7)), **Figure 4** (by (1.8.1) to (1.8.8)), and **Figure 5** (by (1.9.1) to (1.9.9)), for

$$y' = f(x, y) = x^3 y, \quad y(0) = 1. \quad (3.1)$$

(Here, the analytical solution is $y(x) = \exp\left(\frac{x^4}{4}\right)$.)

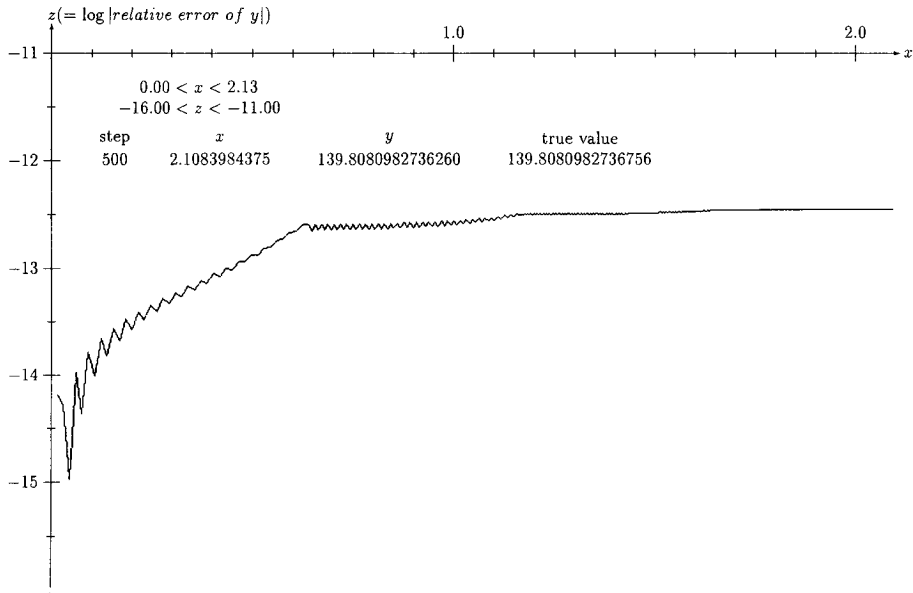


Figure 3 (seven-nodes)
(Starting values: $h = 0.03125$ and $eps = 10^{-12}$)

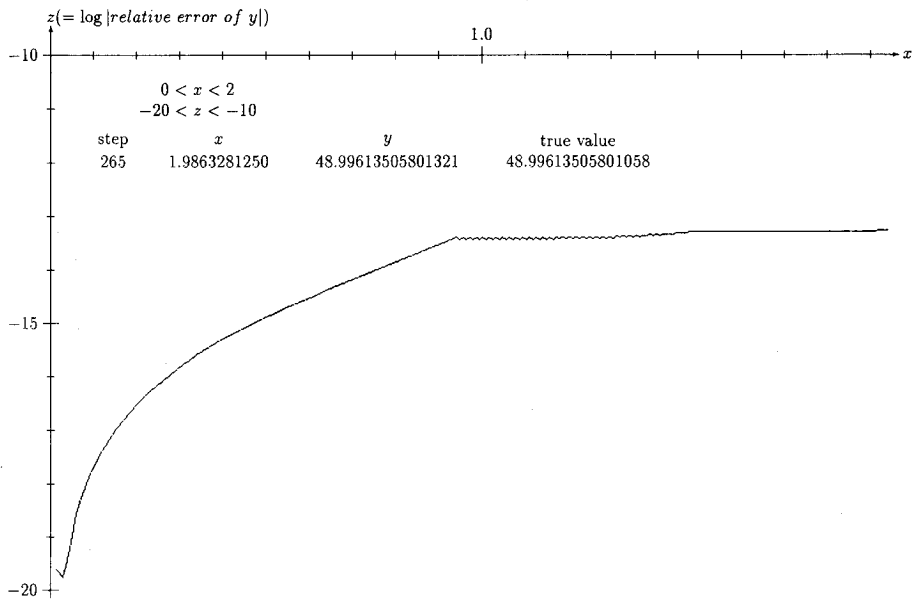


Figure 4 (eight-nodes)
(Starting values: $h = 0.015625$ and $eps = 10^{-15}$)

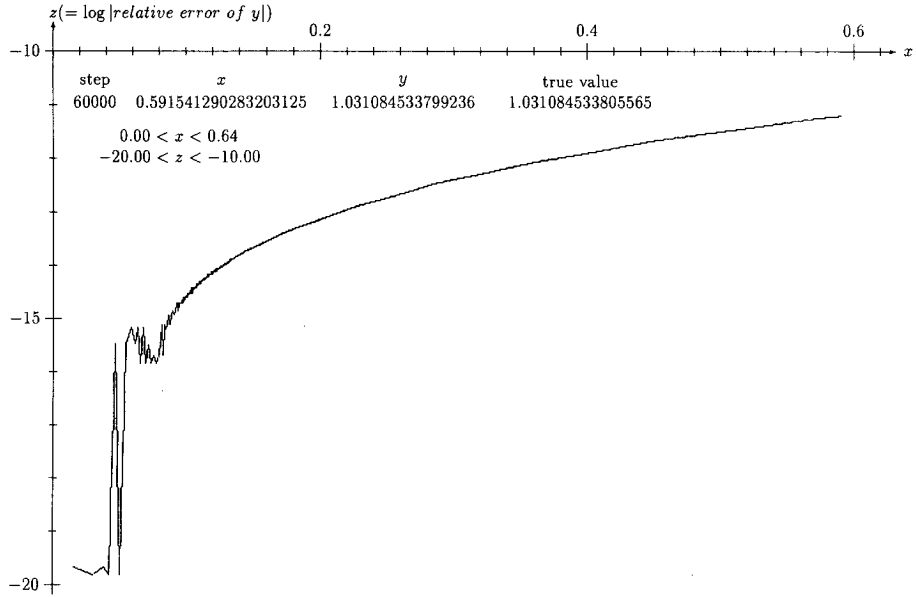


Figure 5 (nine-nodes)
(Starting values: $h = 0.015625$ and $eps = 10^{-15}$)

In each of **Figure 3**, **Figure 4**, and **Figure 5**, the horizontal axis shows the value of x , and the vertical axis indicates the common logarithms of the absolute value of the relative error of y .

4. Numerical example II

Figure 6 (by (1.6.1) to (1.6.6)), **Figure 7** (by (1.7.1) to (1.7.7)), **Table 2-a**, **Table 2-b**, **Table 2-c**, **Table 2-d** (by (1.8.1) to (1.8.8)), which are at the end of this paper, and **Figure 8** (by (1.9.1) to (1.9.9)) have parameters

$$y' = f(x, y) = x^3 y^{-3}, \quad y(0) = 1. \quad (4.1)$$

(Here, the analytical solution is $y(x) = \sqrt[4]{x^4 + 1}$.)

In each of **Figure 6**, **Figure 7**, and **Figure 8**, the horizontal axis indicates the value of x , and the vertical axis shows the common logarithms of the absolute value of the relative error of y .

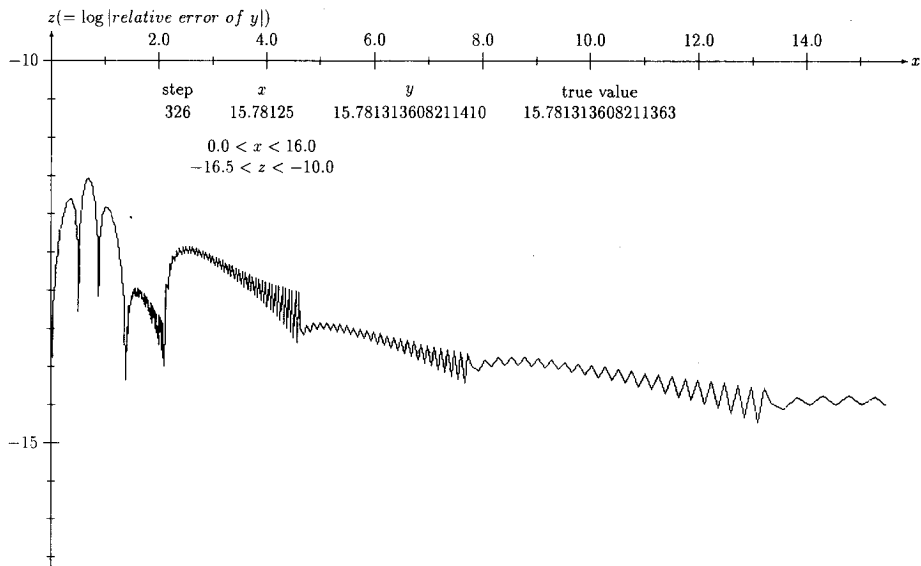


Figure 6 (six-nodes)
 (Starting values: $h = 0.03125$ and $eps = 10^{-10}$)

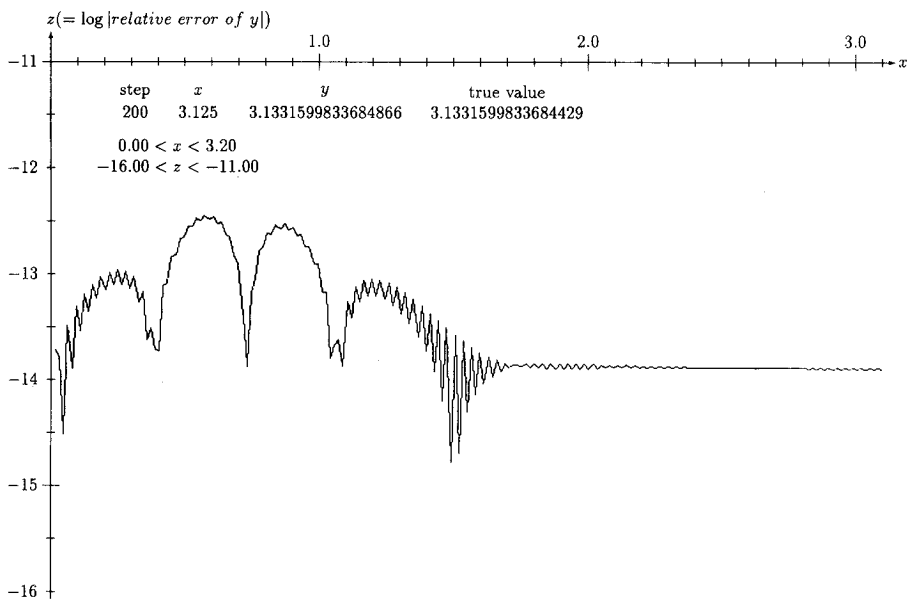


Figure 7 (seven-nodes)
 (Starting values: $h = 0.015625$ and $eps = 10^{-15}$)

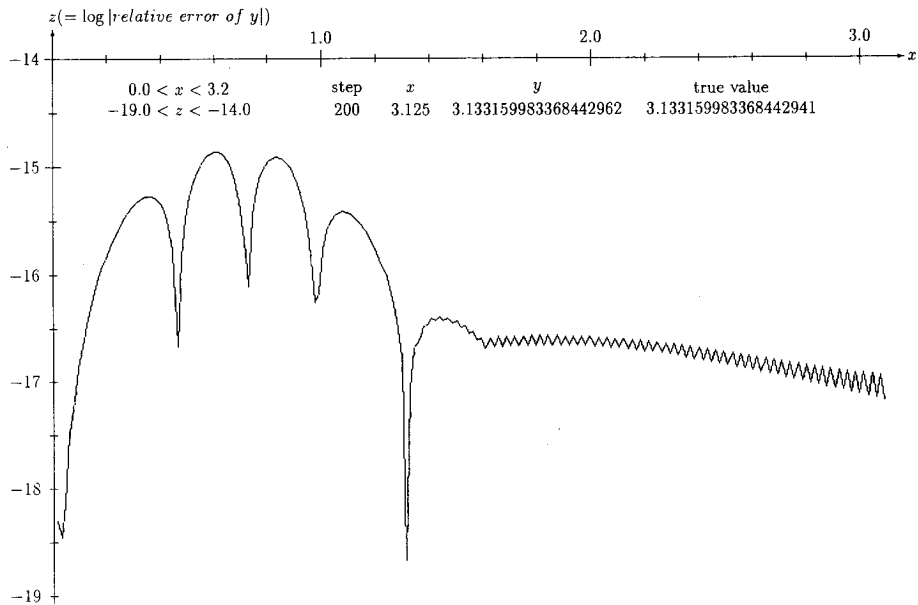


Figure 8 (nine-nodes)
(Starting values: $h = 0.015625$ and $\epsilon = 10^{-10}$)

References

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Table 1-a (six-nodes)
(Starting values: $h = 0.03125$ and $eps = 10^{-10}$)

step	x	y	true value	step	x	y	true value
0	0	1	1	50	1.14062 5	1.52678 88918	1.52678 88924
1	.03125	1.00000 02384	1.00000 02384	51	1.15625	1.56335 60150	1.56335 60156
2	.0625	1.00000 38147	1.00000 38147	52	1.17187 5	1.60236 72275	1.60236 72281
3	.09375	1.00001 93121	1.00001 93121	53	1.1875	1.64400 47105	1.64400 47112
4	.125	1.00006 10370	1.00006 10370	54	1.20312 5	1.68846 71913	1.68846 71920
5	.15625	1.00014 90227	1.00014 90227	55	1.21875	1.73597 17023	1.73597 17031
6	.1875	1.00030 90382	1.00030 90382	56	1.23437 5	1.78675 55531	1.78675 55539
7	.21875	1.00057 26069	1.00057 26069	57	1.25	1.84107 85383	1.84107 85392
8	.25	1.00097 70395	1.00097 70395	58	1.25781 25	1.86965 51447	1.86965 51455
9	.28125	1.00156 54884	1.00156 54884	59	1.26562 5	1.89922 54174	1.89922 54183
10	.3125	1.00238 70302	1.00238 70302	60	1.27343 75	1.92982 93094	1.92982 93102
11	.34375	1.00349 67859	1.00349 67860	61	1.28125	1.96150 86999	1.96150 87009
12	.375	1.00495 60886	1.00495 60886	62	1.28906 25	1.99430 75033	1.99430 75042
13	.40625	1.00683 27101	1.00683 27102	63	1.29687 5	2.02827 17803	2.02827 17813
14	.4375	1.00920 11609	1.00920 11609	64	1.30468 75	2.06344 98607	2.06344 98617
15	.46875	1.01214 30762	1.01214 30763	65	1.3125	2.09989 24695	2.09989 24705
16	.5	1.01574 77085	1.01574 77086	66	1.32031 25	2.13765 28645	2.13765 28655
17	.53125	1.02011 25435	1.02011 25437	67	1.32812 5	2.17678 69797	2.17678 69807
18	.5625	1.02534 40641	1.02534 40643	68	1.33593 75	2.21735 35804	2.21735 35814
19	.59375	1.03155 86878	1.03155 86880	69	1.34375	2.25941 44260	2.25941 44271
20	.625	1.03888 39090	1.03888 39092	70	1.35156 25	2.30303 44451	2.30303 44461
21	.65625	1.04745 96831	1.04745 96834	71	1.35937 5	2.34828 19201	2.34828 19212
22	.6875	1.05744 00948	1.05744 00951	72	1.36718 75	2.39522 86862	2.39522 86872
23	.71875	1.06899 53618	1.06899 53622	73	1.375	2.44395 03405	2.44395 03417
24	.73437 5	1.07542 17528	1.07542 17532	74	1.38281 25	2.49452 64683	2.49452 64694
25	.75	1.08231 42386	1.08231 42390	75	1.39062 5	2.54704 08805	2.54704 08817
26	.76562 5	1.08969 97817	1.08969 97820	76	1.39843 75	2.60158 18710	2.60158 18722
27	.78125	1.09760 68920	1.09760 68924	77	1.40625	2.65824 24875	2.65824 24888
28	.79687 5	1.10606 57506	1.10606 57509	78	1.41406 25	2.71712 08240	2.71712 08252
29	.8125	1.11510 83424	1.11510 83428	79	1.42187 5	2.77832 03299	2.77832 03313
30	.82812 5	1.12476 86038	1.12476 86041	80	1.42968 75	2.84195 01437	2.84195 01450
31	.84375	1.13508 25820	1.13508 25824	81	1.4375	2.90812 54461	2.90812 54475
32	.85937 5	1.14608 86116	1.14608 86120	82	1.44531 25	2.97696 78407	2.97696 78421
33	.875	1.15782 75063	1.15782 75068	83	1.45312 5	3.04860 57585	3.04860 57600
34	.89062 5	1.17034 27705	1.17034 27709	84	1.46093 75	3.12317 48934	3.12317 48948
35	.90625	1.18368 08298	1.18368 08302	85	1.46875	3.20081 86653	3.20081 86669
36	.92187 5	1.19789 12858	1.19789 12862	86	1.47656 25	3.28168 87196	3.28168 87211
37	.9375	1.21302 71942	1.21302 71947	87	1.48437 5	3.36594 54585	3.36594 54601
38	.95312 5	1.22914 53719	1.22914 53723	88	1.49218 75	3.45375 86140	3.45375 86156
39	.96875	1.24630 67330	1.24630 67335	89	1.5	3.54530 78595	3.54530 78612
40	.98437 5	1.26457 66610	1.26457 66614	90	1.50781 25	3.64078 34677	3.64078 34694
41	1	1.28402 54162	1.28402 54167	91	1.51562 5	3.74038 70152	3.74038 70170
42	1.01562 5	1.30472 85875	1.30472 85879	92	1.52343 75	3.84433 21403	3.84433 21421
43	1.03125	1.32676 75884	1.32676 75889	93	1.53125	3.95284 53554	3.95284 53574
44	1.04687 5	1.35023 02074	1.35023 02079	94	1.53906 25	4.06616 69216	4.06616 69235
45	1.0625	1.37521 12146	1.37521 12151	95	1.54687 5	4.18455 17866	4.18455 17886
46	1.07812 5	1.40181 30345	1.40181 30350	96	1.55468 75	4.30827 05962	4.30827 05983
47	1.09375	1.43014 64904	1.43014 64910	97	1.5625	4.43761 07795	4.43761 07817
48	1.10937 5	1.46033 16310	1.46033 16315	98	1.57031 25	4.57287 77197	4.57287 77219
49	1.125	1.49249 86470	1.49249 86476	99	1.57812 5	4.71439 60124	4.71439 60147

Table 1-b (six-nodes)

step	x	y	true value	step	x	y	true value
100	1.58593 75	4.86251 08234	4.86251 08257	150	1.83984 375	17.54201 04889	17.54201 04980
101	1.59375	5.01758 93503	5.01758 93528	151	1.84375	17.97539 70353	17.97539 70447
102	1.60156 25	5.18002 24002	5.18002 24027	152	1.84765 625	18.42235 72068	18.42235 72164
103	1.60937 5	5.35022 60898	5.35022 60925	153	1.85156 25	18.88338 18232	18.88338 18331
104	1.61718 75	5.52864 36826	5.52864 36853	154	1.85546 875	19.35898 15625	19.35898 15726
105	1.625	5.71574 75704	5.71574 75732	155	1.85937 5	19.84968 78413	19.84968 78517
106	1.63281 25	5.91204 14156	5.91204 14185	156	1.86328 125	20.35605 37388	20.35605 37494
107	1.64062 5	6.11806 24657	6.11806 24688	157	1.86718 75	20.87865 49637	20.87865 49746
108	1.64843 75	6.33438 40571	6.33438 40602	158	1.87109 375	21.41809 08700	21.41809 08811
109	1.65625	6.56161 83219	6.56161 83252	159	1.875	21.97498 55198	21.97498 55313
110	1.66406 25	6.80041 91202	6.80041 91236	160	1.87890 625	22.54998 87998	22.54998 88115
111	1.67187 5	7.05148 52135	7.05148 52171	161	1.88281 25	23.14377 75902	23.14377 76023
112	1.67968 75	7.31556 37039	7.31556 37077	162	1.88671 875	23.75705 69931	23.75705 70055
113	1.6875	7.59345 37621	7.59345 37660	163	1.89062 5	24.39056 16194	24.39056 16322
114	1.69531 25	7.88601 06699	7.88601 06740	164	1.89453 125	25.04505 69404	25.04505 69534
115	1.70312 5	8.19415 02075	8.19415 02118	165	1.89843 75	25.72134 07047	25.72134 07182
116	1.70703 125	8.35436 73503	8.35436 73546	166	1.90234 375	26.42024 44280	26.42024 44418
117	1.71093 75	8.51885 34161	8.51885 34206	167	1.90625	27.14263 49539	27.14263 49681
118	1.71484 375	8.68774 20811	8.68774 20856	168	1.91015 625	27.88941 60951	27.88941 61097
119	1.71875	8.86117 17704	8.86117 17750	169	1.91406 25	28.66153 03556	28.66153 03707
120	1.72265 625	9.03928 58460	9.03928 58507	170	1.91796 875	29.45996 07403	29.45996 07557
121	1.72656 25	9.22223 28008	9.22223 28056	171	1.92187 5	30.28573 26546	30.28573 26704
122	1.73046 875	9.41016 64628	9.41016 64677	172	1.92578 125	31.13991 59017	31.13991 59180
123	1.73437 5	9.60324 62060	9.60324 62111	173	1.92968 75	32.02362 67802	32.02362 67970
124	1.73828 125	9.80163 71727	9.80163 71777	174	1.93359 375	32.93803 02892	32.93803 03064
125	1.74218 75	10.00551 05028	10.00551 05081	175	1.9375	33.88434 24454	33.88434 24632
126	1.74609 375	10.21504 35766	10.21504 35819	176	1.94140 625	34.86383 27201	34.86383 27384
127	1.75	10.43042 02641	10.43042 02695	177	1.94531 25	35.87782 66003	35.87782 66191
128	1.75390 625	10.65183 11890	10.65183 11945	178	1.94921 875	36.92770 82832	36.92770 83026
129	1.75781 25	10.87947 40015	10.87947 40072	179	1.95312 5	38.01492 35096	38.01492 35295
130	1.76171 875	11.11355 36652	11.11355 36709	180	1.95703 125	39.14098 25446	39.14098 25651
131	1.76562 5	11.35428 27547	11.35428 27606	181	1.96093 75	40.30746 33139	40.30746 33351
132	1.76953 125	11.60188 17688	11.60188 17748	182	1.96484 375	41.51601 47043	41.51601 47261
133	1.77343 75	11.85657 94550	11.85657 94613	183	1.96875	42.76836 00371	42.76836 00596
134	1.77734 375	12.11861 31513	12.11861 31576	184	1.97265 625	44.06630 07245	44.06630 07476
135	1.78125	12.38822 91402	12.38822 91467	185	1.97656 25	45.41172 01194	45.41172 01433
136	1.78515 625	12.66568 30221	12.66568 30287	186	1.98046 875	46.80658 75696	46.80658 75942
137	1.78906 25	12.95124 01021	12.95124 01089	187	1.98437 5	48.25296 26865	48.25296 27120
138	1.79296 875	13.24517 57975	13.24517 58044	188	1.98828 125	49.75299 98434	49.75299 98696
139	1.79687 5	13.54777 60609	13.54777 60680	189	1.99218 75	51.30895 29128	51.30895 29399
140	1.80078 125	13.85933 78250	13.85933 78322	190	1.99609 375	52.92318 02600	52.92318 02878
141	1.80468 75	14.18016 94661	14.18016 94736	191	2	54.59815 00043	54.59815 00331
142	1.80859 375	14.51059 12901	14.51059 12976	192	2.00390 625	56.33644 55667	56.33644 55964
143	1.8125	14.85093 60392	14.85093 60470	193	2.00781 25	58.14077 15162	58.14077 15469
144	1.81640 625	15.20154 94243	15.20154 94321	194	2.01171 875	60.01395 97372	60.01395 97689
145	1.82031 25	15.56279 06794	15.56279 06876	195	2.01562 5	61.95897 59316	61.95897 59644
146	1.82421 875	15.93503 31448	15.93503 31531	196	2.01953 125	63.97892 64791	63.97892 65128
147	1.82812 5	16.31866 48741	16.31866 48827	197	2.02343 75	66.07706 56730	66.07706 57080
148	1.83203 125	16.71408 92728	16.71408 92815	198	2.02734 375	68.25680 33581	68.25680 33941
149	1.83593 75	17.12172 57642	17.12172 57731	199	2.03125	70.52171 29880	70.52171 30253

Table 2-a (eight-nodes)
(Starting values: $h = 0.015625$ and $eps = 10^{-12}$)

step	x	y	true value
0	0	1	1
1	.01562 5	1.00000 00149 01161	1.00000 00149 01161
2	.03125	1.00000 02384 18494	1.00000 02384 18494
3	.04687 5	1.00000 12069 91871	1.00000 12069 91871
4	.0625	1.00000 38146 75438	1.00000 38146 75438
5	.07812 5	1.00000 93130 95645	1.00000 93130 95645
6	.09375	1.00001 93113 45508	1.00001 93113 45508
7	.10937 5	1.00003 57757 68122	1.00003 57757 68122
8	.125	1.00006 10295 69110	1.00006 10295 69110
9	.14062 5	1.00009 77521 84425	1.00009 77521 84424
10	.15625	1.00014 89783 16824	1.00014 89783 16824
11	.17187 5	1.00021 80965 41513	1.00021 80965 41512
12	.1875	1.00030 88473 69008	1.00030 88473 69007
13	.20312 5	1.00042 53206 41438	1.00042 53206 41438
14	.21875	1.00057 19521 27441	1.00057 19521 27440
15	.23437 5	1.00075 35191 70806	1.00075 35191 70805
16	.25	1.00097 51352 39428	1.00097 51352 39426
17	.26562 5	1.00124 22432 14183	1.00124 22432 14181
18	.28125	1.00156 06072 52658	1.00156 06072 52656
19	.29687 5	1.00193 63030 60397	1.00193 63030 60395
20	.3125	1.00237 57064 03242	1.00237 57064 03239
21	.32812 5	1.00288 54796 98692	1.00288 54796 98689
22	.34375	1.00347 25565 32629	1.00347 25565 32625
23	.35937 5	1.00414 41239 60612	1.00414 41239 60607
24	.375	1.00490 76024 70766	1.00490 76024 70761
25	.39062 5	1.00577 06235 08302	1.00577 06235 08296
26	.40625	1.00674 10044 90256	1.00674 10044 90250
27	.42187 5	1.00782 67212 73178	1.00782 67212 73171
28	.4375	1.00903 58780 76127	1.00903 58780 76120
29	.45312 5	1.01037 66749 06255	1.01037 66749 06248
30	.46875	1.01185 73725 83849	1.01185 73725 83842
31	.48437 5	1.01348 62555 17243	1.01348 62555 17236
32	.5	1.01527 15924 34472	1.01527 15924 34465
33	.51562 5	1.01722 15953 36487	1.01722 15953 36481
34	.53125	1.01934 43769 94771	1.01934 43769 94766
35	.54687 5	1.02164 79073 72333	1.02164 79073 72329
36	.5625	1.02413 99693 99479	1.02413 99693 99477
37	.57812 5	1.02682 81145 82335	1.02682 81145 82333
38	.59375	1.02971 96189 60783	1.02971 96189 60783
39	.60937 5	1.03282 14399 61370	1.03282 14399 61371
40	.625	1.03614 01747 07970	1.03614 01747 07973
41	.64062 5	1.03968 20203 57257	1.03968 20203 57261
42	.65625	1.04345 27370 16188	1.04345 27370 16193
43	.67187 5	1.04745 76137 74300	1.04745 76137 74307
44	.6875	1.05170 14383 44700	1.05170 14383 44707
45	.70312 5	1.05618 84707 54799	1.05618 84707 54807
46	.71875	1.06092 24214 62431	1.06092 24214 62439
47	.73437 5	1.06590 64341 96733	1.06590 64341 96741
48	.75	1.07114 30737 38447	1.07114 30737 38454
49	.76562 5	1.07663 43187 63757	1.07663 43187 63764

Table 2-b (eight-nodes)

step	x	y	true value
50	.78125	1.08238 15597 82367	1.08238 15597 82373
51	.79687 5	1.08838 56021 07318	1.08838 56021 07323
52	.8125	1.09464 66737 04033	1.09464 66737 04036
53	.82812 5	1.10116 44376 82065	1.10116 44376 82067
54	.84375	1.10793 80091 17498	1.10793 80091 17498
55	.85937 5	1.11496 59758 28942	1.11496 59758 28941
56	.875	1.12224 64226 77154	1.12224 64226 77152
57	.89062 5	1.12977 69589 18358	1.12977 69589 18355
58	.90625	1.13755 47481 14892	1.13755 47481 14888
59	.92187 5	1.14557 65400 93604	1.14557 65400 93599
60	.9375	1.15383 87044 41892	1.15383 87044 41887
61	.95312 5	1.16233 72650 42384	1.16233 72650 42378
62	.96875	1.17106 79351 68589	1.17106 79351 68583
63	.98437 5	1.18002 61526 93836	1.18002 61526 93830
64	1	1.18920 71150 02727	1.18920 71150 02721
65	1.01562 5	1.19860 58132 36344	1.19860 58132 36339
66	1.03125	1.20821 70655 47905	1.20821 70655 47900
67	1.04687 5	1.21803 55490 92748	1.21803 55490 92743
68	1.0625	1.22805 58305 24030	1.22805 58305 24026
69	1.07812 5	1.23827 23948 12050	1.23827 23948 12046
70	1.09375	1.24867 96722 49561	1.24867 96722 49557
71	1.10937 5	1.25927 20635 47062	1.25927 20635 47060
72	1.125	1.27004 39629 60134	1.27004 39629 60132
73	1.14062 5	1.28098 97794 25046	1.28098 97794 25044
74	1.15625	1.29210 39557 08909	1.29210 39557 08907
75	1.17187 5	1.30338 09856 06407	1.30338 09856 06406
76	1.1875	1.31481 54292 36791	1.31481 54292 36790
77	1.20312 5	1.32640 19265 12431	1.32640 19265 12430
78	1.21875	1.33813 52088 64115	1.33813 52088 64115
79	1.23437 5	1.35001 01093 18717	1.35001 01093 18716
80	1.25	1.36202 15710 32184	1.36202 15710 32184
81	1.26562 5	1.37416 46543 85435	1.37416 46543 85435
82	1.28125	1.38643 45427 52936	1.38643 45427 52936
83	1.29687 5	1.39882 65470 53979	1.39882 65470 53980
84	1.3125	1.41133 61091 95220	1.41133 61091 95221
85	1.32812 5	1.42395 88045 10175	1.42395 88045 10175
86	1.34375	1.43669 03432 97517	1.43669 03432 97517
87	1.35937 5	1.44952 65715 55275	1.44952 65715 55275
88	1.375	1.46246 34710 02731	1.46246 34710 02731
89	1.39062 5	1.47549 71584 76107	1.47549 71584 76108
90	1.40625	1.48862 38847 78214	1.48862 38847 78214
91	1.42187 5	1.50184 00330 56186	1.50184 00330 56187
92	1.4375	1.51514 21167 75458	1.51514 21167 75458
93	1.45312 5	1.52852 67773 52224	1.52852 67773 52224
94	1.46875	1.54199 07815 00949	1.54199 07815 00949
95	1.48437 5	1.55553 10183 48025	1.55553 10183 48025
96	1.5	1.56914 44963 57498	1.56914 44963 57498
97	1.51562 5	1.58282 83401 09909	1.58282 83401 09909
98	1.53125	1.59657 97869 70753	1.59657 97869 70753
99	1.54687 5	1.61039 61836 80819	1.61039 61836 80819

Table 2-c (eight-nodes)

step	x	y	true value
100	1.5625	1.62427 49828 96791	1.62427 49828 96791
101	1.57812 5	1.63821 37397 06897	1.63821 37397 06897
102	1.59375	1.65221 01081 43160	1.65221 01081 43160
103	1.60937 5	1.66626 18377 08809	1.66626 18377 08809
104	1.625	1.68036 67699 36743	1.68036 67699 36742
105	1.64062 5	1.69452 28349 92544	1.69452 28349 92544
106	1.65625	1.70872 80483 33344	1.70872 80483 33344
107	1.67187 5	1.72298 05074 31948	1.72298 05074 31948
108	1.6875	1.73727 83885 73882	1.73727 83885 73881
109	1.70312 5	1.75161 99437 33536	1.75161 99437 33536
110	1.71875	1.76600 34975 34221	1.76600 34975 34221
111	1.73437 5	1.78042 74442 95778	1.78042 74442 95778
112	1.75	1.79489 02451 72381	1.79489 02451 72380
113	1.76562 5	1.80939 04253 82253	1.80939 04253 82253
114	1.78125	1.82392 65715 30267	1.82392 65715 30267
115	1.79687 5	1.83849 73290 23719	1.83849 73290 23719
116	1.8125	1.85310 13995 81022	1.85310 13995 81021
117	1.84375	1.88240 45540 12601	1.88240 45540 12596
118	1.875	1.91182 66858 90646	1.91182 66858 90641
119	1.90625	1.94135 92030 38255	1.94135 92030 38246
120	1.9375	1.97099 42058 97433	1.9709 94205 897427
121	1.96875	2.00072 44285 90161	2.00072 44285 90151
122	2	2.03054 31848 68937	2.03054 31848 68931
123	2.03125	2.06044 43186 33908	2.06044 43186 33898
124	2.0625	2.09042 21586 90881	2.09042 21586 90875
125	2.09375	2.12047 14774 30274	2.12047 14774 30265
126	2.125	2.15058 74531 17356	2.15058 74531 17350
127	2.15625	2.18076 56354 98871	2.18076 56354 98862
128	2.1875	2.21100 19144 47267	2.21100 19144 47262
129	2.21875	2.24129 24913 81959	2.24129 24913 81950
130	2.25	2.27163 38532 24816	2.27163 38532 24811
131	2.28125	2.30202 27486 65701	2.30202 27486 65693
132	2.3125	2.33245 61665 31083	2.33245 61665 31079
133	2.34375	2.36293 13160 66175	2.36293 13160 66168
134	2.375	2.39344 56089 56674	2.39344 56089 56671
135	2.40625	2.42399 66429 31708	2.42399 66429 31702
136	2.4375	2.45458 21868 03210	2.45458 21868 03208
137	2.46875	2.48520 01668 10391	2.48520 01668 10385
138	2.5	2.51584 86541 49548	2.51584 86541 49546
139	2.53125	2.54652 58535 80896	2.54652 58535 80891
140	2.5625	2.57723 00930 13724	2.57723 00930 13722
141	2.59375	2.60795 98139 80833	2.60795 98139 80828
142	2.625	2.63871 35629 21127	2.63871 35629 21126
143	2.65625	2.66948 99831 97274	2.66948 99831 97269
144	2.6875	2.70028 78077 81807	2.70028 78077 81807
145	2.71875	2.73110 58525 51753	2.73110 58525 51748
146	2.75	2.76194 30101 37060	2.76194 30101 37059
147	2.78125	2.79279 82442 73714	2.79279 82442 73709
148	2.8125	2.82367 05846 16554	2.82367 05846 16554
149	2.84375	2.85455 91219 71523	2.85455 91219 71519

Table 2-d (eight-nodes)

step	x	y	true value
150	2.875	2.88546 30039 10314	2.88546 30039 10314
151	2.90625	2.91638 14307 34368	2.91638 14307 34364
152	2.9375	2.94731 36517 57707	2.94731 36517 57707
153	2.96875	2.97825 89618 81411	2.97825 89618 81407
154	3	3.00921 66984 34563	3.00921 66984 34564
155	3.03125	3.04018 62382 59267	3.04018 62382 59263
156	3.0625	3.07116 69950 18896	3.07116 69950 18896
157	3.09375	3.10215 84167 11128	3.10215 84167 11124
158	3.125	3.13315 99833 68442	3.13315 99833 68443
159	3.15625	3.16417 12049 30868	3.16417 12049 30864
160	3.1875	3.19519 16192 76562	3.19519 16192 76563
161	3.21875	3.22622 07903 97633	3.22622 07903 97629
162	3.25	3.25725 83067 09178	3.25725 83067 09179
163	3.28125	3.28830 37794 81109	3.28830 37794 81106
164	3.3125	3.31935 68413 82692	3.31935 68413 82693
165	3.34375	3.35041 71451 31146	3.35041 71451 31142
166	3.375	3.38148 43622 35856	3.38148 43622 35857
167	3.40625	3.41255 81818 30995	3.41255 81818 30991
168	3.4375	3.44363 83095 89436	3.44363 83095 89437
169	3.46875	3.47472 44667 11981	3.47472 44667 11977
170	3.5	3.50581 63889 85859	3.50581 63889 85860
171	3.53125	3.53691 38259 07542	3.53691 38259 07538
172	3.5625	3.56801 65398 64735	3.56801 65398 64736
173	3.59375	3.59912 43053 73417	3.59912 43053 73413
174	3.625	3.63023 69083 65545	3.63023 69083 65547
175	3.65625	3.66135 41455 24004	3.66135 41455 24001
176	3.6875	3.69247 58236 61018	3.69247 58236 61020
177	3.71875	3.72360 17591 37202	3.72360 17591 37198
178	3.75	3.75473 17773 17992	3.75473 17773 17994
179	3.78125	3.78586 57120 65108	3.78586 57120 65105
180	3.8125	3.81700 34052 60228	3.81700 34052 60229
181	3.84375	3.84814 47063 58929	3.84814 47063 58926
182	3.875	3.87928 94719 72460	3.87928 94719 72462
183	3.90625	3.91043 75654 75708	3.91043 75654 75705
184	3.9375	3.94158 88566 39255	3.94158 88566 39257
185	3.96875	3.97274 32212 84174	3.97274 32212 84171
186	4	4.00390 05409 57702	4.00390 05409 57704
187	4.03125	4.03506 07026 28701	4.03506 07026 28697
188	4.0625	4.06622 35984 01245	4.06622 35984 01247
189	4.09375	4.09738 91252 45464	4.09738 91252 45461
190	4.125	4.12855 71847 44158	4.12855 71847 44160
191	4.15625	4.15972 76828 54484	4.15972 76828 54481
192	4.1875	4.19090 05296 83393	4.19090 05296 83395
193	4.21875	4.22207 56392 76262	4.22207 56392 76258
194	4.25	4.25325 29294 17531	4.25325 29294 17533
195	4.28125	4.28443 23214 42918	4.28443 23214 42915
196	4.3125	4.31561 37400 62126	4.31561 37400 62129
197	4.34375	4.34679 71131 91739	4.34679 71131 91735
198	4.375	4.37798 23717 97296	4.37798 23717 97299
199	4.40625	4.40916 94497 44351	4.40916 94497 44348