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On a New Multistep Method II

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Abstract

In the papers from [1] through [6], we have devised some multistep methods, and we considered the stability of our formulas. In this paper, we want to consider the corrector formulas for a new multistep method, using the Lemma concerning a determinant.

(But, we must leave the predictor formulas to our next paper. The reason is that we need some more time to get good results on the automatic control of the step-size, using our predictors and our correctors.)

By this Lemma, on (m)-node formulas, the corrector formula for y_{n+m-2} has $0\times y'_{n+m-1}$ (the other formulas for $y_k(k=n+1,\ n+2,\ \cdots,\ n+m-3$, and $n+m-1)$ have the term y'_{n+m-1}), when m is an even number. And, the corrector formulas for y_{n+m-1} have the truncation error: $0\times h^{m+1}y^{(m+1)}(\xi)^1+constant\times h^{m+2}y^{(m+2)}(\xi)$ (the other formulas for $y_k(k=n+1,\ n+2,\ \cdots,\ n+m-2)$ have the truncation error: $constant\times h^{m+1}y^{(m+1)}(\xi)$), when m is an odd number.

1. Lemma

Lemma

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n & (2n)^2/2 \\ 1 & 2^2 & 3^2 & 4^2 & \cdots & (2n-1)^2 & (2n)^2 & (2n)^3/3 \\ 1 & 2^3 & 3^3 & 4^3 & \cdots & (2n-1)^3 & (2n)^3 & (2n)^4/4 \\ 1 & 2^4 & 3^4 & 4^4 & \cdots & (2n-1)^4 & (2n)^4 & (2n)^5/5 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2^{2n} & 3^{2n} & 4^{2n} & \cdots & (2n-1)^{2n} & (2n)^{2n} & (2n)^{2n+1}/(2n+1) \\ 1 & 2^{2n+1} & 3^{2n+1} & 4^{2n+1} & \cdots & (2n-1)^{2n+1} & (2n)^{2n+1} & (2n)^{2n+2}/(2n+2) \end{vmatrix}=0$$

Proof

Setting

[1] ξ has a well-qualified value in the concerned interval. (In this paper we use ξ with this meaning after this.)

$$f(x)=\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n & x \\ 1 & 2^2 & 3^2 & 4^2 & \cdots & (2n-1)^2 & (2n)^2 & x^2 \\ 1 & 2^3 & 3^3 & 4^3 & \cdots & (2n-1)^3 & (2n)^3 & x^3 \\ 1 & 2^4 & 3^4 & 4^4 & \cdots & (2n-1)^4 & (2n)^4 & x^4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2^{2n} & 3^{2n} & 4^{2n} & \cdots & (2n-1)^{2n} & (2n)^{2n} & x^{2n} \\ 1 & 2^{2n+1} & 3^{2n+1} & 4^{2n+1} & \cdots & (2n-1)^{2n+1} & (2n)^{2n+1} & x^{2n+1} \end{vmatrix},$$

$$\int f(x)dx=\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n & x^2/2 \\ 1 & 2^2 & 3^2 & 4^2 & \cdots & (2n-1)^2 & (2n)^2 & x^3/3 \\ 1 & 2^3 & 3^3 & 4^3 & \cdots & (2n-1)^3 & (2n)^3 & x^4/4 \\ 1 & 2^4 & 3^4 & 4^4 & \cdots & (2n-1)^4 & (2n)^4 & x^5/5 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2^{2n} & 3^{2n} & 4^{2n} & \cdots & (2n-1)^{2n} & (2n)^{2n} & x^{2n+1}/(2n+1) \\ 1 & 2^{2n+1} & 3^{2n+1} & 4^{2n+1} & \cdots & (2n-1)^{2n+1} & (2n)^{2n+1} & x^{2n+2}/(2n+2) \end{vmatrix}.$$

Here,

$$\begin{aligned} f(x)=(2n)!\times x\times(-1)^{\frac{2n(2n+1)}{2}}&(1-2)(1-3)(1-4)\ldots(1-(2n-1))(1-2n)(1-x) \\ &\times(2-3)(2-4)\ldots(2-(2n-1))(2-2n)(2-x) \\ &\times(3-4)\ldots(3-(2n-1))(3-2n)(3-x) \\ &\times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\times((2n-1)-2n)((2n-1)-x) \\ &\times(2n-x), \end{aligned}$$

where

$$\begin{aligned} &x(x-1)(x-2)(x-3)(x-4)\ldots(x-(2n-1))(x-2n) \\ &=((x-n)+n)((x-n)+n-1)((x-n)+n-2)((x-n)+n-3)\times\ldots \\ &\quad\times((x-n)-(n-1))((x-n)-n). \end{aligned}$$

So, the graph of $y=f(x)$ is symmetric concerning the point $(n, 0)$.
Therefore,

$$\int_0^{2n} f(x)dx=0.$$

Then,

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n & (2n)^2/2 \\ 1 & 2^2 & 3^2 & 4^2 & \cdots & (2n-1)^2 & (2n)^2 & (2n)^3/3 \\ 1 & 2^3 & 3^3 & 4^3 & \cdots & (2n-1)^3 & (2n)^3 & (2n)^4/4 \\ 1 & 2^4 & 3^4 & 4^4 & \cdots & (2n-1)^4 & (2n)^4 & (2n)^5/5 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2^{2n} & 3^{2n} & 4^{2n} & \cdots & (2n-1)^{2n} & (2n)^{2n} & (2n)^{2n+1}/(2n+1) \\ 1 & 2^{2n+1} & 3^{2n+1} & 4^{2n+1} & \cdots & (2n-1)^{2n+1} & (2n)^{2n+1} & (2n)^{2n+2}/(2n+2) \end{vmatrix} = 0$$

2. Formulas

By the same method as the paper [1], we have obtained the corrector formulas for our multistep method. We have used the same predictor formulas as the paper [1], and we have made the new predictor formulas. But, we have not yet acquired good results. In this paper, in k-node formulas, we have made the corrector formulas for y_{n+1}, y_{n+2}, $\cdots$, and y_{n+k-1}, using y_n, y'_n, y'_{n+1}, y'_{n+2}, $\cdots$, and y'_{n+k-1}.

In the cases for $(2m)$-node formulas, the formulas for y_{n+2m-2} has $0\times y'_{n+2m-1}$ (the other formulas for $y_k(k=n+1,\ n+2,\ \cdots,\ n+2m-3$, and $n+2m-1)$ have the term y'_{n+2m-1}).

In the cases for $(2m+1)$-node formulas, the formulas for y_{n+2m} has the truncation error: $0\times h^{2m+2}y^{(2m+2)}(\xi)+constant\times h^{2m+3}y^{(2m+3)}(\xi)$ (the other formulas for $y_k(k=n+1,\ n+2,\ \cdots$, and $n+2m-1)$ have the truncation error: $constant\times h^{2m+2}y^{(2m+2)}(\xi)$).

(1) five-node formulas

(a) correctors

$$y_{n+4}=y_n+\frac{h}{45}(14y'_n+64y'_{n+1}+24y'_{n+2}+64y'_{n+3}+14y'_{n+4})+0\times h^6y^{(6)}(\xi)$$
$$-\frac{8}{945}h^7y^{(7)}(\xi), \tag{2.5.1}$$

$$y_{n+3}=y_n+\frac{h}{80}(27y'_n+102y'_{n+1}+72y'_{n+2}+42y'_{n+3}-3y'_{n+4})+\frac{3}{160}h^6y^{(6)}(\xi), \tag{2.5.2}$$

$$y_{n+2}=y_n+\frac{h}{90}(29y'_n+124y'_{n+1}+24y'_{n+2}+4y'_{n+3}-y'_{n+4})+\frac{1}{90}h^6y^{(6)}(\xi), \tag{2.5.3}$$

$$y_{n+1}=y_n+\frac{h}{720}(251y'_n+646y'_{n+1}-264y'_{n+2}+106y'_{n+3}-19y'_{n+4})+\frac{3}{160}h^6y^{(6)}(\xi), \tag{2.5.4}$$

(2) six-node formulas

(a) correctors

$$y_{n+5}=y_n+\frac{h}{288}(95y'_n+375y'_{n+1}+250y'_{n+2}+250y'_{n+3}+375y'_{n+4}+95y'_{n+5})$$
$$-\frac{275}{12096}h^7y^{(7)}(\xi), \tag{2.6.1}$$

$$y_{n+4}=y_n+\frac{h}{45}(14y'_n+64y'_{n+1}+24y'_{n+2}+64y'_{n+3}+14y'_{n+4}+0\times y'_{n+5})-\frac{8}{945}h^7y^{(7)}(\xi), \tag{2.6.2}$$

$$y_{n+3}=y_n+\frac{h}{160}(51y'_n+219y'_{n+1}+114y'_{n+2}+114y'_{n+3}-21y'_{n+4}+3y'_{n+5})$$
$$-\frac{29}{2240}h^7y^{(7)}(\xi), \tag{2.6.3}$$

$$y_{n+2}=y_n+\frac{h}{90}(28y'_n+129y'_{n+1}+14y'_{n+2}+14y'_{n+3}-6y'_{n+4}+y'_{n+5})-\frac{37}{3780}h^7y^{(7)}(\xi), \tag{2.6.4}$$

$$y_{n+1}=y_n+\frac{h}{1440}(475y'_n+1427y'_{n+1}-798y'_{n+2}+482y'_{n+3}-173y'_{n+4}+27y'_{n+5})$$
$$-\frac{863}{60480}h^7y^{(7)}(\xi), \tag{2.6.5}$$

(3) seven-node formulas
(a) correctors

$$y_{n+6}=y_n+\frac{h}{140}(41y'_n+216y'_{n+1}+27y'_{n+2}+272y'_{n+3}+27y'_{n+4}$$
$$+216y'_{n+5}+41y'_{n+6})+0\times h^8y^{(8)}-\frac{9}{1400}h^9y^{(9)}(\xi), \tag{2.7.1}$$

$$y_{n+5}=y_n+\frac{h}{12096}(3715y'_n+17400y'_{n+1}+6375y'_{n+2}+16000y'_{n+3}+11625y'_{n+4}$$
$$+5640y'_{n+5}-275y'_{n+6})+\frac{275}{24192}h^8y^{(8)}(\xi), \tag{2.7.2}$$

$$y_{n+4}=y_n+\frac{h}{945}(286y'_n+1392y'_{n+1}+384y'_{n+2}+1504y'_{n+3}+174y'_{n+4}$$
$$+48y'_{n+5}-8y'_{n+6})+\frac{8}{945}h^8y^{(8)}(\xi), \tag{2.7.3}$$

$$y_{n+3}=y_n+\frac{h}{2240}(685y'_n+3240y'_{n+1}+1161y'_{n+2}+2176y'_{n+3}-729y'_{n+4}$$
$$+216y'_{n+5}-29y'_{n+6})+\frac{9}{896}h^8y^{(8)}(\xi), \tag{2.7.4}$$

$$y_{n+2}=y_n+\frac{h}{3780}(1139y'_n+5640y'_{n+1}+33y'_{n+2}+1328y'_{n+3}-807y'_{n+4}$$
$$+264y'_{n+5}-37y'_{n+6})+\frac{8}{945}h^8y^{(8)}(\xi), \tag{2.7.5}$$

$$y_{n+1}=y_n+\frac{h}{60480}(19087y'_n+65112y'_{n+1}-46461y'_{n+2}+37504y'_{n+3}-20211y'_{n+4}$$
$$+6312y'_{n+5}-863y'_{n+6})+\frac{275}{24192}h^8y^{(8)}(\xi), \tag{2.7.6}$$

(4) eight-node formulas
(a) correctors

$$y_{n+7}=y_n+\frac{h}{17280}(5257y'_n+25039y'_{n+1}+9261y'_{n+2}+20923y'_{n+3}+20923y'_{n+4}$$
$$+9261y'_{n+5}+25039y'_{n+6}+5257y'_{n+7})-\frac{8183}{518400}h^9y^{(9)}(\xi), \tag{2.8.1}$$

$$y_{n+6}=y_n+\frac{h}{140}(41y'_n+216y'_{n+1}+27y'_{n+2}+272y'_{n+3}+27y'_{n+4}$$

$$+216y'_{n+5}+41y'_{n+6}+0\times y'_{n+7})-\frac{9}{1400}h^9y^{(9)}(\xi), \tag{2.8.2}$$

$$y_{n+5}=y_n+\frac{h}{24192}(7155y'_n+36725y'_{n+1}+6975y'_{n+2}+41625y'_{n+3}+13625y'_{n+4}$$
$$+17055y'_{n+5}-2475y'_{n+6}+275y'_{n+7})-\frac{175}{20736}h^9y^{(9)}(\xi), \tag{2.8.3}$$

$$y_{n+4}=y_n+\frac{h}{945}(278y'_n+1448y'_{n+1}+216y'_{n+2}+1784y'_{n+3}-106y'_{n+4}$$
$$+216y'_{n+5}-64y'_{n+6}+8y'_{n+7})-\frac{107}{14175}h^9y^{(9)}(\xi), \tag{2.8.4}$$

$$y_{n+3}=y_n+\frac{h}{4480}(1325y'_n+6795y'_{n+1}+1377y'_{n+2}+5927y'_{n+3}-3033y'_{n+4}$$
$$+1377y'_{n+5}-373y'_{n+6}+45y'_{n+7})-\frac{369}{44800}h^9y^{(9)}(\xi), \tag{2.8.5}$$

$$y_{n+2}=y_n+\frac{h}{3780}(1107y'_n+5864y'_{n+1}-639y'_{n+2}+2448y'_{n+3}-1927y'_{n+4}$$
$$+936y'_{n+5}-261y'_{n+6}+32y'_{n+7})-\frac{119}{16200}h^9y^{(9)}(\xi), \tag{2.8.6}$$

$$y_{n+1}=y_n+\frac{h}{120960}(36799y'_n+139849y'_{n+1}-121797y'_{n+2}+123133y'_{n+3}-88547y'_{n+4}$$
$$+41499y'_{n+5}-11351y'_{n+6}+1375y'_{n+7})-\frac{33953}{3628800}h^9y^{(9)}(\xi), \tag{2.8.7}$$

(5) nine-node formulas
(a) correctors

$$y_{n+8}=y_n+\frac{h}{14175}(3956y'_n+23552y'_{n+1}-3712y'_{n+2}+41984y'_{n+3}$$
$$-18160y'_{n+4}+41984y'_{n+5}-3712y'_{n+6}+23552y'_{n+7}+3956y'_{n+8})$$
$$+0\times h^{10}y^{(10)}-\frac{2368}{467775}h^{11}y^{(11)}(\xi), \tag{2.9.1}$$

$$y_{n+7}=y_n+\frac{h}{518400}(149527y'_n+816634y'_{n+1}+48706y'_{n+2}+1085938y'_{n+3}$$
$$+54880y'_{n+4}+736078y'_{n+5}+522046y'_{n+6}+223174y'_{n+7}-8183y'_{n+8})$$
$$+\frac{8183}{1036800}h^{10}y^{(10)}(\xi). \tag{2.9.2}$$

$$y_{n+6}=y_n+\frac{h}{1400}(401y'_n+2232y'_{n+1}+18y'_{n+2}+3224y'_{n+3}-360y'_{n+4}$$
$$+2664y'_{n+5}+158y'_{n+6}+72y'_{n+7}-9y'_{n+8})+\frac{9}{1400}h^{10}y^{(10)}(\xi), \tag{2.9.3}$$

$$y_{n+5}=y_n+\frac{h}{145152}(41705y'_n+230150y'_{n+1}+7550y'_{n+2}+318350y'_{n+3}-4000y'_{n+4}$$
$$+170930y'_{n+5}-49150y'_{n+6}+11450y'_{n+7}-1225y'_{n+8})+\frac{25}{3584}h^{10}y^{(10)}(\xi), \tag{2.9.4}$$

$$y_{n+4}=y_n+\frac{h}{14175}(4063y'_n+22576y'_{n+1}+244y'_{n+2}+32752y'_{n+3}-9080y'_{n+4}$$
$$+9232y'_{n+5}-3956y'_{n+6}+976y'_{n+7}-107y'_{n+8})+\frac{94}{14175}h^{10}y^{(10)}(\xi), \tag{2.9.5}$$

$$y_{n+3}=y_n+\frac{h}{44800}(12881y'_n+70902y'_{n+1}+3438y'_{n+2}+79934y'_{n+3}-56160y'_{n+4}$$
$$+34434y'_{n+5}-14062y'_{n+6}+3402y'_{n+7}-369y'_{n+8})+\frac{25}{3584}h^{10}y^{(10)}(\xi), \quad (2.9.6)$$

$$y_{n+2}=y_n+\frac{h}{113400}(32377y'_n+182584y'_{n+1}-42494y'_{n+2}+120088y'_{n+3}-116120y'_{n+4}$$
$$+74728y'_{n+5}-31154y'_{n+6}+7624y'_{n+7}-833y'_{n+8})+\frac{9}{1400}h^{10}y^{(10)}(\xi), \quad (2.9.7)$$

$$y_{n+1}=y_n+\frac{h}{3628800}(1070017y'_n+4467094y'_{n+1}-4604594y'_{n+2}+5595358y'_{n+3}$$
$$-5033120y'_{n+4}+3146338y'_{n+5}-1291214y'_{n+6}+312874y'_{n+7}-33953y'_{n+8})$$
$$+\frac{8183}{1036800}h^{10}y^{(10)}(\xi) \quad (2.9.8)$$

3. Our procedure

We make the starting values from the initial values, using the Taylor's expansions or other methods.

In the cases for $(2m+1)$-node formulas, we have the truncation error: $0\times h^{2m+2}y^{(2m+2)}(\xi)+constant\times h^{2m+3}y^{(2m+3)}(\xi)$ for the corrector of y_{n+2m}, and the truncation errors: $constant\times h^{2m+2}y^{(2m+2)}(\xi)$ for the correctors from y_{n+1} through y_{n+2m-1}. In the cases for $(2m)$-node formulas, we have the truncation errors: $constant\times h^{2m+1}y^{(2m+1)}(\xi)$ for all correctors.

By this reason, we use our correctors in the cases for $(2m+1)$-node. In the cases for $(2m)$-node formulas, we must use the multistep method used in the paper [5], from the reason of the small truncation error.

References

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