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On the Ten-Node Interpolation Using Taylor Expansions

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Abstract

When we solve the initial value problem of ordinary differential equations, if we attempt to cut the step size in half using the ten-node central value interpolation formula, we find that it is not possible to cut the five nodes at the top and the bottom in half. The purpose of this paper is to produce formulas for the five nodes at the top and the bottom using Taylor expansions.

The errors at the top and the bottom are bigger than the errors at the center. Therefore, at the top and the bottom it is preferable to use the twelve-node interpolation formula, presented in [2]

1. Introduction

Using the Predictor-Corrector method, we have been studying the initial value problem for first order ordinary differential equations. Iterations of the corrector are terminated when the difference between the new value and the previous value becomes less than the previously fixed value. As the iterated values start to deviate from the true value, we need halve the step size, after the number of iteration is more than the previously fixed value. So we used the fourth order Bessel central difference formula to cut the step size in half. However, the big error remains after corrections by the corrector. For this reason, we want to make the high order interpolation formula, using the Taylor expansion.

In Section 2, we explain our point of view. In Section 3, we explain how to save effort by using some cancellations. In Section 4, we will develop our method. In Section 5, we consider the ralation between "the positive direction k and the negative direction $10-k$ formula" and "the positive direction $10-k$ and the negative direction k formula". In Sections 6 to 11, various cases of interpolation formulas are obtained. In Section 12 and Table 1, we summarize the values for finish side formulas. The values for top side formulas are summarized in Table 2, and the values of our interpolation formulas are summarized in Table 3.

2. A method of obtaining the interpolation formula

When we consider a linear combination of $f((2i-1)h/2), i=1,2,\ldots,4; i=0,-1,-2,\ldots,-5$, we denote by 10^{+4}_{-6} the interpolation formula obtained from positive direction 4 nodes and negative direction 6 nodes. This will be described below.

The Taylor expansion for

$$c_1 f(h/2)+c_2 f(-h/2)+c_3 f(3h/2)+c_4 f(-3h/2)+c_5 f(5h/2)+c_6 f(-5h/2)$$

$+c_7f(7h/2)+c_8f(-7h/2)+c_9f(-9h/2)+c_{10}f(-11h/2)$

(here c_i are coefficients of the linear combination,)
is equal to

$$(\sum_{i=1}^{10} c_i)f(0)+(c_1-c_2+3(c_3-c_4)+5(c_5-c_6)+7(c_7-c_8)-9c_9-11c_{10})hf'(0)/2$$

$$+(c_1+c_2+3^2(c_3+c_4)+5^2(c_5+c_6)+7^2(c_7+c_8)+9^2c_9+11^2c_{10})h^2f''(0)/(2!\times 2^2)$$

$$+(c_1-c_2+3^3(c_3-c_4)+5^3(c_5-c_6)+7^3(c_7-c_8)-9^3c_9-11^3c_{10})h^3f^{(3)}(0)/(3!\times 2^3)$$

$$+(c_1+c_2+3^4(c_3+c_4)+5^4(c_5+c_6)+7^4(c_7+c_8)+9^4c_9+11^4c_{10})h^4f^{(4)}(0)/(4!\times 2^4)$$

$$+(c_1-c_2+3^5(c_3-c_4)+5^5(c_5-c_6)+7^5(c_7-c_8)-9^5c_9-11^5c_{10})h^5f^{(5)}(0)/(5!\times 2^5)$$

$$+(c_1+c_2+3^6(c_3+c_4)+5^6(c_5+c_6)+7^6(c_7+c_8)+9^6c_9+11^6c_{10})h^6f^{(6)}(0)/(6!\times 2^6)$$

$$+(c_1-c_2+3^7(c_3-c_4)+5^7(c_5-c_6)+7^7(c_7-c_8)-9^7c_9-11^7c_{10})h^7f^{(7)}(0)/(7!\times 2^7)$$

$$+(c_1+c_2+3^8(c_3+c_4)+5^8(c_5+c_6)+7^8(c_7+c_8)+9^8c_9+11^8c_{10})h^8f^{(8)}(0)/(8!\times 2^8)$$

$$+(c_1-c_2+3^9(c_3-c_4)+5^9(c_5-c_6)+7^9(c_7-c_8)-9^9c_9-11^9c_{10})h^9f^{(9)}(0)/(9!\times 2^9)$$

$$+(c_1+c_2+3^{10}(c_3+c_4)+5^{10}(c_5+c_6)+7^{10}(c_7+c_8)+9^{10}c_9+11^{10}c_{10})h^{10}f^{(10)}(0)/(10!\times 2^{10})$$

$$+\cdots.$$

Here, we set

$$\sum_{i=1}^{10} c_i = 1,\ \sum_{i=1}^{4}(2i-1)^j(c_{2i-1}+(-1)^jc_{2i})+(-1)^j(9^jc_9+11^jc_{10}) = 0, j = 1,2,\ldots,9.$$

Then we obtain the following 10^{+4}_{-6} interpolation formula:

$$f(0) = c_1f(h/2)+c_2f(-h/2)+c_3f(3h/2)+c_4f(-3h/2)+c_5f(5h/2)+c_6f(-5h/2)$$

$$+c_7f(7h/2)+c_8f(-7h/2)+c_9f(-9h/2)+c_{10}f(-11h/2).$$

Here, the error term is the term for $f^{(10)}(0)$. In this way, we obtain ten-node interpolation formulas.

3. The 10^{+5}_{-5} formula, using 5 nodes in the positive direction and 5 nodes in the negative direction

From the Taylor expansion :

$$y(ih/2) = y(0)+(ih/2)y'(0)+(ih/2)^2y''(0)/2!+\ldots, i = -9,-7,-5,-3,-1,1,3,5,7,9,$$

we obtain

$$y(ih/2)+y(-ih/2) = 2(y(0)+(ih/2)^2y''(0)/2!+(ih/2)^4y^{(4)}(0)/4!+\cdots).$$

Here, we have no odd order term. So, we obtain the following equation:

$$c_5f(-9h/2)+c_4f(-7h/2)+c_3f(-5h/2)+c_2f(-3h/2)+c_1f(-h/2)$$

$$+c_1f(h/2)+c_2f(3h/2)+c_3f(5h/2)+c_4f(7h/2)+c_5f(9h/2)$$

$$=2(\sum_{i=1}^{5}c_i)f(0)+2(c_1+3^2c_2+5^2c_3+7^2c_4+9^2c_5)(h/2)^2f''(0)/2!$$

$$+2(c_1+3^4c_2+5^4c_3+7^4c_4+9^4c_5)(h/2)^4f^{(4)}(0)/4!+2(c_1+3^6c_2+5^6c_3+7^6c_4+9^6c_5)(h/2)^6f^{(6)}(0)/6!$$

$$+2(c_1+3^8c_2+5^8c_3+7^8c_4+9^8c_5)(h/2)^8f^{(8)}(0)/8!+2(c_1+3^{10}c_2+5^{10}c_3+7^{10}c_4+9^{10}c_5)(h/2)^{10}f^{(10)}(0)/10!$$

$$+\cdots.$$

Solving the following equations:

$$2(c_1+c_2+c_3+c_4+c_5)=1,\quad c_1+3^2c_2+5^2c_3+7^2c_4+9^2c_5=0,$$

$$c_1+3^4c_2+5^4c_3+7^4c_4+9^4c_5=0,\quad c_1+3^6c_2+5^6c_3+7^6c_4+9^6c_5=0,$$

$$c_1+3^8c_2+5^8c_3+7^8c_4+9^8c_5=0,$$

we have the following values:

$$c_1=39690/2^{16},\quad c_2=-8820/2^{16},\quad c_3=2268/2^{16}\quad c_4=-405/2^{16}\quad c_5=35/2^{16}.$$

Using these $c_i, i=1,2,\ldots,5$, we obtain the following interpolation formula:

$$f(0)=(39690(f(h/2)+f(-h/2))-8820(f(3h/2)+f(-3h/2))+2268(f(5h/2)+f(-5h/2))$$

$$-405(f(7h/2)+f(-7h/2))+35(f(9h/2)+f(-9h/2)))/65536,$$

where the error term is

$$-2(c_1+3^{10}c_2+5^{10}c_3+7^{10}c_4+9^{10}c_5)\frac{h^{10}}{10!\times 2^{10}}f^{(10)}(\xi)$$

$$=-3^2*7h^{10}f^{(10)}(\xi)/2^{18}=-63h^{10}f^{(10)}(\xi)/262144.$$

4. The 10^{+4}_{-6} formula, using 4 nodes in the positive direction and 6 nodes in the negative direction

From the Taylor expansion, we have the following equation:

$$c_{10}f(-11h/2)+c_9f(-9h/2)+c_8f(-7h/2)+c_6f(-5h/2)+c_4f(-3h/2)$$

$$+c_2f(-h/2)+c_1f(h/2)+c_3f(3h/2)+c_5f(5h/2)+c_7f(7h/2)$$

$$=(\sum_{i=1}^{10}c_i)f(0)+(c_1-c_2+3(c_3-c_4)+5(c_5-c_6)+7(c_7-c_8)-9c_9-11c_{10})(h/2)f'(0)$$

$+(c_1+c_2+3^2(c_3+c_4)+5^2(c_5+c_6)+7^2(c_7+c_8)+9^2c_9+11^2c_{10})(h/2)^2f''(0)/2!$

$+(c_1-c_2+3^3(c_3-c_4)+5^3(c_5-c_6)+7^3(c_7-c_8)-9^3c_9-11^3c_{10})(h/2)^3f^{(3)}(0)/3!$

$+(c_1+c_2+3^4(c_3+c_4)+5^4(c_5+c_6)+7^4(c_7+c_8)+9^4c_9+11^4c_{10})(h/2)^4f^{(4)}(0)/4!+\cdots$

$+(c_1-c_2+3^9(c_3-c_4)+5^9(c_5-c_6)+7^9(c_7-c_8)-9^9c_9-11^9c_{10})(h/2)^9f^{(9)}(0)/9!$

$+(c_1+c_2+3^{10}(c_3+c_4)+5^{10}(c_5+c_6)+7^{10}(c_7+c_8)+9^{10}c_9+11^{10}c_{10})(h/2)^2f^{(10)}(0)/10!+\cdots.$

Solving the following simultaneous equations:

$\sum_{i=1}^{10} c_i = 1,\ c_1-c_2+3(c_3-c_4)+5(c_5-c_6)+7(c_7-c_8)-9c_9-11c_{10} = 0,$

$c_1+c_2+3^2(c_3+c_4)+5^2(c_5+c_6)+7^2(c_7+c_8)+9^2c_9+11^2c_{10} = 0,$

$c_1-c_2+3^3(c_3-c_4)+5^3(c_5-c_6)+7^3(c_7-c_8)-9^3c_9-11^3c_{10} = 0,\ldots,$

$c_1+c_2+3^8(c_3+c_4)+5^8(c_5+c_6)+7^8(c_7+c_8)+9^8c_9+11^8c_{10} = 0,$

$c_1-c_2+3^9(c_3-c_4)+5^9(c_5-c_6)+7^9(c_7-c_8)-9^9c_9-11^9c_{10} = 0,$

we have the following values:

$c_1 = 32340/2^{16},\ c_2 = 48510/2^{16},\ c_3 = -4620/2^{16},\ c_4 = -16170/2^{16},\ c_5 = 693/2^{16},$

$c_6 = 6468/2^{16},\ c_7 = -55/2^{16},\ c_8 = -1980/2^{16},\ c_9 = 385/2^{16},\ c_{10} = -35/2^{16}.$

From these $c_i, i = 1,2,\ldots,10$, we obtain the following interpolation formula:

$f(0) = (32340f(h/2)+48510f(-h/2)-4620f(3h/2)-16170f(-3h/2)+693f(5h/2)+6468f(-5h/2)$

$-55f(7h/2)-1980f(-7h/2)+385f(-9h/2)-35f(-11h/2))/65536,$
and the error term:

$-(c_1+c_2+3^{10}(c_3+c_4)+5^{10}(c_5+c_6)+7^{10}(c_7+c_8)+9^{10}c_9+11^{10}c_{10})h^{10}f^{(10)}(\xi)/(10!\times 2^{10})$

$= 7*11h^{10}f^{(10)}(\xi)/2^{18} = 77h^{10}f^{(10)}(\xi)/262144.$

5. The 10^{+6}_{-4} formula, using 6 nodes in the positive direction and 4 nodes in the negative direction

In the same way in Section 4, i.e, the 10^{+4}_{-6} formula, we get the following simultaneous equations:

$\sum_{i=1}^{10} c_i = 1,\ c_1-c_2+3(c_3-c_4)+5(c_5-c_6)+7(c_7-c_8)+9c_9+11c_{10} = 0,$

$c_1+c_2+3^2(c_3+c_4)+5^2(c_5+c_6)+7^2(c_7+c_8)+9^2c_9+11^2c_{10} = 0,$

$c_1-c_2+3^3(c_3-c_4)+5^3(c_5-c_6)+7^3(c_7-c_8)+9^3c_9+11^3c_{10}=0,\ldots\ldots,$

$c_1-c_2+3^9(c_3-c_4)+5^9(c_5-c_6)+7^9(c_7-c_8)+9^9c_9+11^9c_{10}=0.$

Here, for $1\le k\le 4$, the $10^{+k}_{-(10-k)}$ formula and the $10^{+(10-k)}_{-k}$ formula have same

$\sum_{i=1}^{k}(2i-1)^{2j}(c_{2i-1}+c_{2i})+(2k+1)^{2j}c_{2k+1}+\cdots+(19-2k)^{2j}c_{10}=0, j=0,1,\ldots,4,$

and on the other hand, in the $10^{+k}_{-(10-k)}$ formula we have

$\sum_{i=1}^{k}(2i-1)^{2j-1}(c_{2i-1}-c_{2i})-(2k+1)^{2j-1}c_{2k+1}-\cdots-(19-2k)^{2j-1}c_{10}=0, j=1,2,\ldots,5,$

but in the $10^{+(10-k)}_{-k}$ formula we get

$\sum_{i=1}^{k}(2i-1)^{2j-1}(c_{2i-1}-c_{2i})+(2k+1)^{2j-1}c_{2k+1}+\cdots+(19-2k)^{2j-1}c_{10}=0, j=1,2,\ldots,5,$

After eliminating $(c_{2i-1}-c_{2i}), i=1,2,\ldots,k$, the equations that determine the coefficients of the $10^{+k}_{-(10-k)}$ formula have the form $ac_{2k+1}+\cdots+ec_{10}=0$. Similarly, the equations for the $10^{+(10-k)}_{-k}$ formula become $-ac_{2k+1}-\cdots-ec_{10}=0$. The right hand side is equal to zero, so these two equations are equal.

Each interpolation formula has the same $c_{2k+1},\ldots,c_{10}$, and $c_{2i-1}, i=1,2,\ldots,k$, replace each other with $c_{2i}, i=1,2,\ldots,k$, respectively.

From these reasons, we have the following values:

$c_1=48510/2^{16},\ c_2=32340/2^{16},\ c_3=-16170/2^{16},\ c_4=-4620/2^{16},\ c_5=6468/2^{16},$

$c_6=693/2^{16},\ c_7=-1980/2^{16},\ c_8=-55/2^{16},\ c_9=385/2^{16},\ c_{10}=-35/2^{16}.$

From these $c_i, i=1,2,\ldots,10$, we obtain the following interpolation formula:

$f(0)=(48510f(h/2)+32340f(-h/2)-16170f(3h/2)-4620f(-3h/2)+6468f(5h/2)$

$+693f(-5h/2)-1980f(7h/2)-55f(-7h/2))+385f(9h/2)-35f(11h/2))/65536,$

and the error term : $77h^{10}f(10)(\xi)/2^{18}$.

6. The 10^{+3}_{-7} formula, using 3 nodes in the positive direction and 7 nodes in the negative direction

In the same way in Section 4, we have the following equations:

$\sum_{i=1}^{10}c_i=1,\ c_1-c_2+3(c_3-c_4)+5(c_5-c_6)-7c_7-9c_8-11c_9-13c_{10}=0$

$c_1+c_2+3^2(c_3+c_4)+5^2(c_5+c_6)+7^2c_7+9^2c_8+11^2c_9+13^2c_{10}=0,\ldots,$

$c_1-c_2+3^9(c_3-c_4)+5^9(c_5-c_6)-7^9c_7-9^9c_8-11^9c_9-13^9c_{10}=0.$

Solving these equations, we get the following values:

$c_1 = 25740/2^{16},\ c_2 = 60060/2^{16},\ c_3 = -2145/2^{16},\ c_4 = -30030/2^{16},\ c_5 = 143/2^{16},$

$c_6 = 18018/2^{16},\ c_7 = -8580/2^{16},\ c_8 = 2860/2^{16},\ c_9 = -585/2^{16},\ c_{10} = 55/2^{16}.$

Using these values, we obtain the following formula:

$f(0) = (25740f(h/2)+60060f(-h/2)-2145f(3h/2)-30030f(-3h/2)+143f(5h/2)$

$+18018f(-5h/2)-8580f(-7h/2)+2860f(-9h/2)-585f(-11h/2)+55f(-13h/2))/65536,$
and the error term: $-11*13h^{10}f^{(10)}(\xi)/2^{18} = -143h^{10}f^{(10)}(\xi)/262144.$

7. The 10^{+7}_{-3} formula, using 7 nodes in the positive direction and 3 nodes in the negative direction

From Section 6, i.e., the 10^{+3}_{-7} formula, we obtain the following values :

$c_1 = 60060/2^{16},\ c_2 = 25740/2^{16},\ c_3 = -30030/2^{16},\ c_4 = -2145/2^{16},\ c_5 = 18018/2^{16},$

$c_6 = 143/2^{16},\ c_7 = -8580/2^{16},\ c_8 = 2860/2^{16},\ c_9 = -585/2^{16},\ c_{10} = 55/2^{16}.$

Thus we have the following interpolation formula:

$f(0) = (60060f(h/2)+25740f(-h/2)-30030f(3h/2)-2145f(-3h/2)+18018f(5h/2)$

$+143f(-5h/2)-8580f(7h/2)+2860f(9h/2)-585f(11h/2)+55f(13h/2))/65536,$

and the error term : $-143h^{10}f^{(10)}(\xi)/2^{18}.$

8. The 10^{+2}_{-8} formula, using 2 nodes in the positive direction and 8 nodes in the negative direction

In the same way in Section 4, we get the following equations:

$\sum_{i=1}^{10} c_i = 1,\ c_1-c_2+3(c_3-c_4)-5c_5-7c_6-9c_7-11c_8-13c_9-15c_{10} = 0,$

$c_1+c_2+3^2(c_3+c_4)+5^2c_5+7^2c_6+9^2c_7+11^2c_8+13^2c_9+15^2c_{10} = 0,$

$c_1-c_2+3^3(c_3-c_4)-5^3c_5-7^3c_6-9^3c_7-11^3c_8-13^3c_9-15^3c_{10} = 0,$

$c_1+c_2+3^4(c_3+c_4)+5^4c_5+7^4c_6+9^4c_7+11^4c_8+13^4c_9+15^4c_{10} = 0,\ldots\ldots,$

$c_1-c_2+3^9(c_3-c_4)-5^9c_5-7^9c_6-9^9c_7-11^9c_8-13^9c_9-15^9c_{10} = 0.$

Solving these equations, we have the following values:

$c_1 = 19305/2^{16},\ c_2 = 77220/2^{16},\ c_3 = -715/2^{16},\ c_4 = -60060/2^{16},\ c_5 = 54054/2^{16},$

$c_6 = -38610/2^{16},\ c_7 = 20020/2^{16},\ c_8 = -7020/2^{16},\ c_9 = 1485/2^{16},\ c_{10} = -143/2^{16}.$

From these values, we have the following interpolation formula:

$$f(0)=(19305f(h/2)+77220f(-h/2)-715f(3h/2)-60060f(-3h/2)+54054f(-5h/2)$$

$$-38610f(-7h/2)+20020f(-9h/2)-7020f(-11h/2)+1485f(-13h/2)-143f(-15h/2))/65536,$$

and the error term is equql to $3*11*13h^{10}f^{(10)}(\xi)/2^{18}=429h^{10}f^{(10)}(\xi)/262144$.

9. The 10^{+8}_{-2} formula, using 8 nodes in the positive direction and 2 nodes in the negative direction

From Section 8, i.e., 10^{+2}_{-8} formula, we obtain the following values:

$c_1=77220/2^{16},\ c_2=19305/2^{16},\ c_3=-60060/2^{16},\ c_4=-715/2^{16},\ c_5=54054/2^{16},$

$c_6=-38610/2^{16},\ c_7=20020/2^{16},\ c_8=-7020/2^{16},\ c_9=1485/2^{16},\ c_{10}=-143/2^{16},$

the following interpolation formula:

$$f(0)=(77220f(h/2)+19305f(-h/2)-60060f(3h/2)-715f(-3h/2)+54054f(5h/2)$$

$$-38610f(7h/2)+20020f(9h/2)-7020f(11h/2)+1485f(13h/2)-143f(15h/2))/65536,$$

and the error term : $429h^{10}f^{(10)}(\xi)/2^{18}$.

10. The 10^{+1}_{-9} formula, using 1 node in the positive direction and 9 nodes in the negative direction

In the same way in Section 4, we have the following equations:

$\sum_{i=1}^{10} c_i=1,\ c_1-c_2-3c_3-5c_4-7c_5-9c_6-11c_7-13c_8-15c_9-17c_{10}=0,$

$c_1+c_2+3^2c_3+5^2c_4+7^2c_5+9^2c_6+11^2c_7+13^2c_8+15^2c_9+17^2c_{10}=0,$

$c_1-c_2-3^3c_3-5^3c_4-7^3c_5-9^3c_6-11^3c_7-13^3c_8-15^3c_9-17^3c_{10}=0,$

$c_1+c_2+3^4c_3+5^4c_4+7^4c_5+9^4c_6+11^4c_7+13^4c_8+15^4c_9+17^4c_{10}=0,\ldots\ldots,$

$c_1-c_2-3^9c_3-5^9c_4-7^9c_5-9^9c_6-11^9c_7-13^9c_8-15^9c_9-17^9c_{10}=0,$

the values:

$c_1=12155/2^{16},\ c_2=109395/2^{16},\ c_3=-145860/2^{16},\ c_4=204204/2^{16},\ c_5=-218790/2^{16},$

$c_6=170170/2^{16},\ c_7=-92820/2^{16},\ c_8=33660/2^{16},\ c_9=-7293/2^{16},\ c_{10}=715/2^{16},$

the following interpolation formula:

$$f(0)=(12155f(h/2)+109395f(-h/2)-145860f(-3h/2)+204204f(-5h/2)-218790f(-7h/2)$$

$+170170f(-9h/2)-92820f(-11h/2)+33660f(-13h/2)-7293f(-15h/2)+715f(-17h/2))/65536,$

and the error term : $-11*13*17h^{10}f^{(10)}(\xi)/2^{18} = -2431h^{10}f^{(10)}(\xi)/262144.$

11. The 10^{+9}_{-1} formula, using 9 nodes in the positive direction and 1 node in the negative direction

From Section 10, i.e., the 10^{+1}_{-9} formula, we obtain the following values:

$c_1 = 109395/2^{16},\ c_2 = 12155/2^{16},\ c_3 = -145860/2^{16},\ c_4 = 204204/2^{16},\ c_5 = -218790/2^{16},$

$c_6 = 170170/2^{16},\ c_7 = -92820/2^{16},\ c_8 = 33660/2^{16},\ c_9 = -7293/2^{16},\ c_{10} = 715/2^{16},$

the following interpolation formula:

$f(0) = (109395f(h/2)+12155f(-h/2)-145860f(3h/2)+204204f(5h/2)-218790f(7h/2)$

$+170170f(9h/2)-92820f(11h/2)+33660f(13h/2)-7293f(15h/2)+715f(17h/2))/65536,$

and the error term: $-2431h^{10}f^{(10)}(\xi)/2^{18}$.

12. An example

In Table 1, Table 2, and Table 3, the odd numbers x, y are obtained from a numerical solution of $y' = x^3y$, the initial value $y(0) = 1$, the step size $h = 0.0625$, and the analytical solution is $y = exp(x^4/4)$. i is the number of iterations, y_t is the value of the analytical solution, and $E_y = (y-y_t)\times 10^{10}$.

In Table 1, y^{+4}_{-6} is obtained from the 10^{+4}_{-6} formula, $E^{+4}_{-6} = (y^{+4}_{-6}-y_t)\times 10^{10}$, y^{+3}_{-7} is obtained from the 10^{+3}_{-7} formula, $E^{+3}_{-7} = (y^{+3}_{-7}-y_t)\times 10^{10}$, y^{+2}_{-8} is obtained from the 10^{+2}_{-8} formula, $E^{+2}_{-8} = (y^{+2}_{-8}-y_t)\times 10^{10}$, y^{+1}_{-9} is obtained from the 10^{+1}_{-9} formula, and $E^{+1}_{-9} = (y^{+1}_{-9}-y_t)\times 10^{10}$.

In Table 2, y^{+9}_{-1} is obtained from the 10^{+9}_{-1} formula, $E^{+9}_{-1} = (y^{+9}_{-1}-y_t)\times 10^{10}$, y^{+8}_{-2} is obtained from the 10^{+8}_{-2} formula, $E^{+8}_{-2} = (y^{+8}_{-2}-y_t)\times 10^{10}$, y^{+7}_{-3} is obtained from the 10^{+7}_{-3} formula, $E^{+7}_{-3} = (y^{+7}_{-3}-y_t)\times 10^{10}$, y^{+6}_{-4} is obtained from the 10^{+6}_{-4} formula, and $E^{+6}_{-4} = (y^{+6}_{-4}-y_t)\times 10^{10}$.

In Table 3, the value y_i are the interpolated values, and $E_i = (y_i-y_t)\times 10^{10}$. y_i in Step No.2 of Step Column is obtained using the 10^{+9}_{-1} interpolation formula. Step No.4, 6, and 8 are obtained using the 10^{+8}_{-2}, 10^{+7}_{-3}, and 10^{+6}_{-4} interpolation formulas respectively. No.36, 38, 40, and 42 are obtained from the 10^{+4}_{-6}, 10^{+3}_{-7}, 12^{+2}_{-8}, and 10^{+1}_{-9}, interpolation formulas respectively. No.10, 12, ..., 32, 34 are obtained from the 10^{+5}_{-5} formula.

References

[1] F. Tamari, R. Tsukamoto, R. Furuki, and H. Yanagiwara, On a New Multistep Method Ⅱ, Bull. of Fukuoka Univ. of Ed. part Ⅲ **49** (2000) 1-6.
[2] F. Tamari and H. Yanagiwara, On the Twelve-Node Interpolation Using Taylor Expansions, Bull. ol Fukuoka Univ. of Ed. part Ⅲ **57** (2008) 53-62.

Table 1

step	x	i	y	E_y	y_t	y_{-6}^{+4}	E_{-6}^{+4}	y_{-7}^{+3}	E_{-7}^{+3}	y_{-8}^{+2}	E_{-8}^{+2}	y_{-9}^{+1}	E_{-9}^{+1}
1	0.3125	2	1.0023870302	0	1.0023870302								
2	0.375	3	1.0049560887	1	1.0049560886								
3	0.4375	3	1.0092011610	1	1.0092011609								
4	0.5	3	1.0157477087	1	1.0157477086								
5	0.5625	3	1.0253440645	2	1.0253440643								
6	0.625	3	1.0388839094	2	1.0388839092								
7	0.65625				1.0474596834	1.0474596835	1						
8	0.6875	4	1.0574400954	3	1.0574400951								
9	0.71875				1.0689953622	1.0689953624	2	1.0689953629	7				
10	0.75	4	1.0823142396	6	1.0823142390								
11	0.78125				1.0976068924	1.0976068927	3	1.0976068935	11	1.0976068922	-2		
12	0.8125	4	1.1151083437	9	1.1151083428								
13	0.84375				1.1350825824	1.1350825829	5	1.1350825843	19	1.1350825822	-2	1.1350825886	62
14	0.875	4	1.1578275083	15	1.1578275068								
15	0.90625				1.1836808302	1.1836808311	9	1.1836808334	32	1.1836808299	-3	1.1836808403	101
16	0.9375	4	1.2130271973	26	1.2130271947								
17	0.96875				1.2463067335	1.2463067350	15	1.2463067389	54	1.2463067331	-4	1.2463067504	169
18	1.0	5	1.2840254212	45	1.2840254167								
19	1.03125				1.3267675889	1.3267675916	27	1.3267675985	96	1.3267675884	-5	1.3267676177	288
20	1.0625	5	1.3752112232	81	1.3752112151								
21	1.09375				1.4301464910	1.4301464959	49	1.4301465082	172	1.4301464903	-7	1.4301465411	501
22	1.125	5	1.4924986622	146	1.4924986476								
23	1.15625				1.5633560156	1.5633560245	89	1.5633560472	316	1.5633560151	-5	1.5633561045	889
24	1.1875	6	1.6440047384	272	1.6440047112								
25	1.21875				1.7359717031	1.7359717195	164	1.7359717622	591	1.7359717031	-0	1.7359718640	1609
26	1.25	6	1.8410785908	516	1.8410785392								
27	1.28125				1.9615087009	1.9615087319	310	1.96150881340	1131	1.9615087030	21	1.9615089986	2977
28	1.3125	6	2.0998925708	1003	2.0998924705								
29	1.34375				2.2594144271	2.2594144871	600	2.2594146488	2217	2.2594144354	83	2.2594149905	5634
30	1.375	7	2.4439505415	1998	2.4439503417								
31	1.40625				2.6582424888	2.6582426078	1190	2.6582429345	4457	2.6582425142	254	2.6582435811	10923
32	1.4375	7	2.9081258559	4084	2.9081254475								
33	1.46875				3.2008186669			3.2008195869	9200	3.2008187374	705	3.2008208388	21719
34	1.5	8	3.5453087189	8577	3.5453078612								
35	1.53125				3.9528453574					3.9528455455	1881	3.9528497927	44353
36	1.5625	8	4.4376126351	18534	4.4376107817								
37	1.59375				5.0175893528							5.0175986702	93174
38	1.625	9	5.7157516997	41265	5.7157475732								

Table 2

step	x	i	y	E_y	y_t	y_{-1}^{+9}	E_{-1}^{+9}	y_{-2}^{+8}	E_{-2}^{+8}	y_{-3}^{+7}	E_{-3}^{+7}	y_{-4}^{+6}	E_{-4}^{+6}
1	0.3125	2	1.0023870302	0	1.0023870302								
2	0.34375				1.0034967860	1.0034967896	36						
3	0.375	3	1.0049560887	1	1.0049560886								
4	0.40625				1.0068327102	1.0068327160	58	1.0068327096	-6				
5	0.4375	3	1.0092011610	1	1.0092011609								
6	0.46875				1.0121430763	1.0121430857	94	1.0121430753	-10	1.0121430766	3		
7	0.5	3	1.0157477087	1	1.0157477086								
8	0.53125				1.0201125437	1.0201125594	157	1.0201125421	-16	1.0201125442	5	1.0201125437	-0
9	0.5625	3	1.0253440645	2	1.0253440643								
10	0.59375				1.0315586880	1.0315587146	266	1.0315586853	-27	1.0315586887	7	1.0315586879	-1
11	0.625	3	1.0388839094	2	1.0388839092								
12	0.65625				1.0474596834	1.0474597295	461	1.0474596788	-46	1.0474596847	13	1.0474596833	-1
13	0.6875	4	1.0574400954	3	1.0574400951								
14	0.71875				1.0689953622	1.0689954436	814	1.0689953542	-80	1.0689953643	21	1.0689953621	-1
15	0.75	4	1.0823142396	6	1.0823142390								
16	0.78125				1.0976068924	1.0976070390	1466	1.0976068781	-143	1.0976068960	36	1.0976068921	-3
17	0.8125	4	1.1151083437	9	1.1151083428								
18	0.84375				1.1350825824	1.1350828522	2698	1.1350825566	-258	1.1350825888	64	1.1350825819	-5
19	0.875	4	1.1578275083	15	1.1578275068								
20	0.90625				1.1836808302	1.1836813376	5074	1.1836807825	-477	1.1836808416	114	1.1836808292	-10
21	0.9375	4	1.2130271973	26	1.2130271947								
22	0.96875				1.2463067335	1.2463077101	9766	1.2463066433	-902	1.2463067543	208	1.2463067315	-20
23	1.0	5	1.2840254212	45	1.2840254167								
24	1.03125				1.3267675889	1.3267695157	19268	1.3267674144	-1745	1.3267676277	388	1.3267675850	-39
25	1.0625	5	1.3752112232	81	1.3752112151								
26	1.09375				1.4301464910	1.4301503922	39012	1.4301461449	-3461	1.4301465652	742	1.4301464832	-78
27	1.125	5	1.4924986622	146	1.4924986476								
28	1.15625				1.5633560156			1.5633553116	-7040	1.5633561611	1455	1.5633559994	-162
29	1.1875	6	1.6440047384	272	1.6440047112								
30	1.21875				1.7359717031					1.7359719956	2925	1.7359716689	-342
31	1.25	6	1.8410785908	516	1.8410785392								
32	1.28125				1.9615087009							1.9615086269	-740
33	1.3125	6	2.0998925708	1003	2.0998924705								
34	1.375	7	2.4439505415	1998	2.4439503417								
35	1.4375	7	2.9081258559	4084	2.9081254475								
36	1.5	8	3.5453087189	8577	3.5453078612								
37	1.5625	8	4.4376126351	18534	4.4376107817								
38	1.625	9	5.7157516997	41265	5.7157475732								

Table 3

step	x	i	y	E_y	y_t	y_i	E_i
1	0.3125	2	1.0023870302	0	1.0023870302		
2	0.34375				1.0034967860	1.0034967896	36
3	0.375	3	1.0049560887	1	1.0049560886		
4	0.40625				1.0068327102	1.0068327096	-6
5	0.4375	3	1.0092011610	1	1.0092011609		
6	0.46875				1.0121430763	1.0121430766	3
7	0.5	3	1.0157477087	1	1.0157477086		
8	0.53125				1.0201125437	1.0201125437	-0
9	0.5625	3	1.0253440645	2	1.0253440643		
10	0.59375				1.0315586880	1.0315586882	2
11	0.625	3	1.0388839094	2	1.0388839092		
12	0.65625				1.0474596834	1.0474596838	4
13	0.6875	4	1.0574400954	3	1.0574400951		
14	0.71875				1.0689953622	1.0689953629	7
15	0.75	4	1.0823142396	6	1.0823142390		
16	0.78125				1.0976068924	1.0976068935	11
17	0.8125	4	1.1151083437	9	1.1151083428		
18	0.84375				1.1350825824	1.1350825844	20
19	0.875	4	1.1578275083	15	1.1578275068		
20	0.90625				1.1836808302	1.1836808336	34
21	0.9375	4	1.2130271973	26	1.2130271947		
22	0.96875				1.2463067335	1.2463067394	59
23	1.0	5	1.2840254212	45	1.2840254167		
24	1.03125				1.3267675889	1.3267675995	106
25	1.0625	5	1.3752112232	81	1.3752112151		
26	1.09375				1.4301464910	1.4301465103	193
27	1.125	5	1.4924986622	146	1.4924986476		
28	1.15625				1.5633560156	1.5633560517	361
29	1.1875	6	1.6440047384	272	1.6440047112		
30	1.21875				1.7359717031	1.7359717717	686
31	1.25	6	1.8410785908	516	1.8410785392		
32	1.28125				1.9615087009	1.9615088348	1339
33	1.3125	6	2.0998925708	1003	2.0998924705		
34	1.34375				2.2594144271	2.2594146951	2680
35	1.375	7	2.4439505415	1998	2.4439503417		
36	1.40625				2.6582424888	2.6582426078	1190
37	1.4375	7	2.9081258559	4084	2.9081254475		
38	1.46875				3.2008186669	3.2008185869	-800
39	1.5	8	3.5453087189	8577	3.5453078612		
40	1.53125				3.9528453574	3.9528455455	1881
41	1.5625	8	4.4376126351	18534	4.4376107817		
42	1.59375				5.0175893528	5.0175986702	93174
43	1.625	9	5.7157516997	41265	5.7157475732		