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SOME PROPERTIES OF FACTORIZED LIE ALGEBRAS

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Abstract. Let $L = A+B$ be a factorized Lie algebra by two subalgebras A and B. For a subalgebra H of L we introduce its factorizer $X(H)$ and investigate some basic properties of $X(H)$. We also prove that if A satisfies the maximal (resp. minimal) condition on ideals and B satisfies the maximal (resp. minimal) condition on subalgebras, then L satisfies the maximal (resp. minimal) condition on ideals.

Introduction

Factorized groups G by two subgroups A and B, i.e. $G = AB$, have been investigated by many authors. Some interesting results were established. For example, Itô [4] proved that if both A and B are abelian, then G is metabelian. Kegel [5] and Wielandt [6] also proved that if G is finite and both A and B are nilpotent, then G is soluble.

As in group theory we can consider the concept of factorized Lie algebras L by two subalgebras A and B, i.e. $L = A+B$. Factorized Lie algebras have been studied for some decades. In [3] we proved that if L is a serially finite Lie algebra over a field of characteristic $\neq 2$ and both A and B are locally nilpotent, then L is locally soluble. We also proved that if L is a serially finite Lie algebra over a field of characteristic zero, then any common serial subalgebra of A and B is serial in L.

In this paper, following [2], we shall state some properties of factorized subalgebras, factorizers of subalgebra and finiteness conditions of factorized Lie algebras.

The main results are as follows: Let $L = A+B$ be a Lie algebra factorized by A and B. If I is an ideal of L, then

(1) $X(I) = (A+I) \cap (B+I)$,
(2) $X(I) = A \cap (B+I)+I = B \cap (A+I)+I = A \cap (B+I)+B \cap (A+I)$,
(3) the vector space $X(I)/A \cap B$ is finite-dimensional, whenever I is finite-dimensional,

where $X(I)$ is the factorizer of I, that is, the smallest factorized subalgebra of L containing I (Proposition 5). Furthermore, let Δ be any of the relations $\leq$, $\lhd$, si, $\lhd^{\alpha}$, asc and assume that A is a Δ-subalgebra of L. If $A \in$ Max-$\lhd$ (resp. Min-$\lhd$) and $B \in$ Max-Δ (resp. Min-Δ), then $L \in$ Max-$\lhd$ (resp. Min-$\lhd$) (Theorem 8).

1. Notation and terminology

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [1] for the use of notation and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For a totally ordered set Σ, a series from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma \mid \sigma \in \Sigma\}$ of subalgebras of L such that

(1) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,
(2) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,
(3) $L \backslash H = \cup_{\sigma \in \Sigma}(\Lambda_\sigma \backslash V_\sigma)$,
(4) $V_\sigma \lhd \Lambda_\sigma$ for all $\sigma \in \Sigma$.

H is a serial subalgebra of L, which we denote by $H \operatorname{ser} L$, if there exists a series from H to L. For an ordinal σ, H is a σ-step ascendant subalgebra of L, denoted by $H \lhd^{\sigma} L$, if there exists an ascending chain $\{H_\alpha\}_{\alpha \leq \sigma}$ of subalgebras of L such that

(1) $H_0 = H$ and $H_\sigma = L$,
(2) $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \sigma$,
(3) $H_\lambda = \cup_{\alpha < \lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant subalgebra of L, denoted by $H \operatorname{asc} L$, if $H \lhd^{\sigma} L$ for some ordinal σ. When σ is finite, H is a subideal of L and denoted by $H \operatorname{si} L$. For an ordinal α, we denote by L^{α} the α-th term of the transfinite lower central series of L.

Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any of the relations $\leq$, $\lhd$, si, $\lhd^{\alpha}$, asc, ser. A Lie algebra L is said to lie in $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra H of L such that $X \subseteq H \,\Delta\, L$. In particular we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. When $L \in \mathrm{L}\mathfrak{X}$ (resp. $\mathrm{L}(\mathrm{ser})\mathfrak{X}$), L is called a locally (resp. a serially) $\mathfrak{X}$-algebra. We write Max-Δ (resp. Min-Δ) for the classes of Lie algebras satisfying the maximal (resp. minimal) condition on Δ-subalgebras. $\mathfrak{F}$, $\mathfrak{A}$, $\mathfrak{N}$ and $\mathrm{E}\mathfrak{A}$ are the classes of Lie algebras which are finite-dimensional, abelian, nilpotent and soluble respectively.

2. Factorized subalgebras

Definition. Let L be a Lie algebra and let A, B be subalgebras of L. As in groups we say that L is *factorized* by A and B if $L = A + B$.

Let $L = A + B$ be a Lie algebra factorized by A and B. Then for an ideal I of L we have

$$L/I = (A+I)/I + (B+I)/I.$$

Hence L/I is factorized by $(A+I)/I$ and $(B+I)/I$. On the other hand, in general, for a subalgebra H of $L = A + B$ it is possible that

$$H \supsetneq (A \cap H) + (B \cap H).$$

Therefore we begin with the following

Lemma 1. *Let $L = A + B$ be a Lie algebra factorized by A and B. For a subalgebra H of L the following conditions are equivalent:*

(i) *If $a + b \in H$ with $a \in A$ and $b \in B$, then a, $b \in H$.*
(ii) *$H = (A \cap H) + (B \cap H)$ and $A \cap B \leq H$.*

Proof. (i) $\Rightarrow$ (ii): It is trivial that $H \supseteq (A \cap H) + (B \cap H)$. If $x = a + b \in H$ with $a \in A$ and $b \in B$, then $a \in A \cap H$ and $b \in B \cap H$ by (i). Therefore $H \subseteq (A \cap H) + (B \cap H)$. Moreover, if $x \in A \cap B$, then we have $x+(-x)=0\in H$ with $x\in A$ and $-x\in B$. By using (i) we obtain $x\in H$. Thus $A\cap B\leq H$.

(ii) $\Rightarrow$ (i): If $a+b\in H$ with $a\in A$ and $b\in B$, then we get $a+b=a_1+b_1$ for some $a_1\in A\cap H$ and $b_1\in B\cap H$ by (ii). Since $a-a_1=b_1-b\in A\cap B\subseteq H$, we have $a=a_1+(a-a_1)\in H$, hence $b=(a+b)-a\in H$. □

Definition. A subalgebra H of a factorized Lie algebra $L=A+B$ is said to be *factorized* in L if it satisfies one of the equivalent conditions of Lemma 1.

The following lemma gives some easy examples of factorized subalgebras.

Lemma 2. *Let $L=A+B$ be a Lie algebra factorized by A and B. Then*

(1) *Every subalgebra H of L which contains A or B is factorized in L.*

(2) *If $A\cap B=0$, then every subalgebra H of L which is contained in A or B is factorized in L.*

Proof. (1) We may suppose that $A\leq H$. Then it follows from the modular law that

$$H=H\cap(A+B)=A+H\cap B=H\cap A+H\cap B.$$

As $A\cap B\leq A\leq H$, we assert that H is factorized in L.

(2) is obvious. □

The following two lemmas give further basic properties of factorized subalgebras.

Lemma 3. *Let $L=A+B$ be a Lie algebra factorized by A and B. Then the following hold.*

(1) *If H_λ is a factorized subalgebra of L for any $\lambda\in\Lambda$, then $\cap_{\lambda\in\Lambda}H_\lambda$ is factorized in L.*

(2) *If H_λ is a factorized ideal of L for any $\lambda\in\Lambda$, then $\Sigma_{\lambda\in\Lambda}H_\lambda$ is factorized in L.*

(3) *Let I be an ideal of L and H a subalgebra of L which contains I. Then H/I is factorized in L/I if and only if H is factorized in L.*

Proof. (1) If $a+b\in\cap_{\lambda\in\Lambda}H_\lambda$ with $a\in A$ and $b\in B$, then we have $a+b\in H_\lambda$ for any $\lambda\in\Lambda$. Since H_λ is factorized in L, we get $a,\ b\in H_\lambda$ for any $\lambda\in\Lambda$, that is, $a,\ b\in\cap_{\lambda\in\Lambda}H_\lambda$.

(2) Put $H=\Sigma_{\lambda\in\Lambda}H_\lambda$. For any $x\in H$, there are $\lambda_1,\dots,\lambda_n\in\Lambda$ such that $x\in H_{\lambda_1}+\cdots+H_{\lambda_n}$. As H_{λ_i} is factorized in L, it follows from Lemma 1 that

$$\begin{aligned}\sum_{i=1}^{n}H_{\lambda_i}&=\sum_{i=1}^{n}((A\cap H_{\lambda_i})+(B\cap H_{\lambda_i}))\\&=\sum_{i=1}^{n}(A\cap H_{\lambda_i})+\sum_{i=1}^{n}(B\cap H_{\lambda_i})\subseteq(A\cap H)+(B\cap H).\end{aligned}$$

Therefore $H\subseteq(A\cap H)+(B\cap H)$. It is trivial that $A\cap B\leq H$.

(3) $\Leftarrow$: Assume that $(a+I)+(b+I)=a+b+I\in H/I$ with $a+I\in(A+I)/I$ and $b+I\in(B+I)/I$ $(a\in A,\ b\in B)$. Then we get $a+b\in H$ with $a\in A$ and $b\in B$. As H is factorized in L, it follows that $a+I,\ b+I\in H/I$.

$\Rightarrow$: Assume that $a+b\in H$ with $a\in A$ and $b\in B$. Then $(a+I)+(b+I)=a+b+I\in H/I$ with $a+I\in(A+I)/I$ and $b+I\in(B+I)/I$. Since H/I is factorized in L/I, we have $a+I,\ b+I\in H/I$. Therefore $a,\ b\in H$. □

Lemma 4. *Let $L = A+B$ be a Lie algebra factorized by A and B. If a subalgebra H is factorized in L, then $H = (A+H) \cap (B+H)$.*

Proof. It is clear that $H \subseteq (A+H) \cap (B+H)$. If $x \in (A+H) \cap (B+H)$, then there are $a \in A$, $b \in B$, u, $v \in H$ such that $x = a+u = b+v$. Therefore we obtain $a+(-b) = v-u \in H$ with $a \in A$ and $-b \in B$. Since H is factorized in L, we get a, $b \in H$, so $x = a+u \in H$. Hence $H \supseteq (A+H) \cap (B+H)$. □

3. Factorizer

Definition. Let $L = A+B$ be a Lie algebra factorized by A and B and let H be a subalgebra of L. By Lemma 3 (1) the intersection $X(H)$ of all factorized subalgebras of L containing H is the smallest factorized subalgebra of L containing H. The subalgebra $X(H)$ is called the *factorizer* of H in $L = A+B$.

The following proposition is one of the main results in the paper, and indicates that the factorizer $X(H)$ has some interesting properties when H is an ideal of L.

Proposition 5. *Let $L = A+B$ be a Lie algebra factorized by A and B, and let K be a permutable subalgebra of L with A and B. Then*

(1) $X(K) = (A+K) \cap (B+K)$.

(2) $X(K) = A \cap (B+K)+K = B \cap (A+K)+K = A \cap (B+K)+B \cap (A+K)$.

(3) *If K is finite-dimensional, then the quotient vector space $X(K)/A \cap B$ is finite-dimensional.*

Proof. (1) Since $A \leq A+K$, $B \leq B+K$, the subalgebras $A+K$ and $B+K$ are factorized in L by Lemma 2 (1). Hence Lemma 3 (1) shows that $(A+K) \cap (B+K)$ is factorized in L containing K. Hence we have $X(K) \leq (A+K) \cap (B+K)$. Conversely if H is a factorized subalgebra of L containing K, then we obtain

$$H = (A+H) \cap (B+H) \geq (A+K) \cap (B+K)$$

by using Lemma 4. Therefore we get $X(K) \geq (A+K) \cap (B+K)$.

(2) By virtue of (1), we have

$$X(K) = (A+K) \cap (B+K) = \begin{cases} A \cap (B+K)+K \\ (A+K) \cap B+K \end{cases}$$

by using the modular law. Moreover, we have

$$A \cap X(K) = A \cap (A+K) \cap (B+K) = A \cap (B+K),$$
$$B \cap X(K) = B \cap (A+K) \cap (B+K) = B \cap (A+K).$$

As $X(K)$ is factorized in L, we get

$$X(K) = A \cap X(K)+B \cap X(K) = A \cap (B+K)+B \cap (A+K).$$

(3) From (2) we obtain $X(K) = A \cap (B+K)+K$. Since

$$X(K)/A \cap (B+K) = (A \cap (B+K)+K)/A \cap (B+K)$$
$$\cong K/A \cap K \quad \text{(as vector spaces)}$$

and K is finite-dimensional, we have $\dim X(K)/A \cap (B+K) < \infty$. As $B+K$ is factorized in L, we get

$$\begin{aligned} B+K &= A\cap(B+K)+B\cap(B+K) \\ &= A\cap(B+K)+B. \end{aligned}$$

Therefore we have

$$\begin{aligned} K/B\cap K &\cong (B+K)/B \\ &= (A\cap(B+K)+B)/B \\ &\cong A\cap(B+K)/A\cap(B+K)\cap B \\ &= A\cap(B+K)/A\cap B \quad \text{(as vector spaces)}. \end{aligned}$$

Hence we obtain dim $A\cap(B+K)/A\cap B<\infty$. Thus we conclude

$$\dim X(K)/A\cap B<\infty. \qquad \square$$

The following result contains a special case that both A and B are abelian.

Proposition 6. *Let $L=A+B$ be a Lie algebra factorized by A and B. If I is an ideal of L such that $A^2\leq I$ and $B^2\leq I$, then $[X(I),L]\subseteq I$, in particular $X(I)$ is also an ideal of L.*

Proof. From Proposition 5 it follows that $X(I)=(A+I)\cap(B+I)$. Therefore we have

$$\begin{aligned} [X(I),L] &= [(A+I)\cap(B+I),A+B] \\ &= [(A+I)\cap(B+I),A]+[(A+I)\cap(B+I),B] \\ &\subseteq [A+I,A]+[B+I,B]\subseteq I. \end{aligned} \qquad \square$$

Here we shall give two examples of infinite-dimensional factorized Lie algebras.

Example. Let $\mathfrak{gl}(n,\mathfrak{k})$ be the general linear Lie algebra of all $n\times n$ matrices over a field $\mathfrak{k}$, and let f_{mn} be the natural embedding from $\mathfrak{gl}(n,\mathfrak{k})$ to $\mathfrak{gl}(m,\mathfrak{k})$ for $n\leq m$. We define

$$L=\varinjlim \mathfrak{gl}(n,\mathfrak{k})=\bigcup_{n\geq 2}\mathfrak{gl}(n,\mathfrak{k}) \quad \text{(the direct limit of } \{\mathfrak{gl}(n,\mathfrak{k}),f_{mn}\}).$$

Further, let $\mathfrak{sl}(n,\mathfrak{k})$ be the special linear Lie algebra of all $n\times n$ matrices with trace zero over $\mathfrak{k}$ and we define

$$S=\bigcup_{n\geq 2}\mathfrak{sl}(n,\mathfrak{k}).$$

Then we have $S\lhd L$ since $L^2\subseteq S$.

(1) Let $\mathfrak{ut}(n,\mathfrak{k})$ be the subalgebra of $\mathfrak{gl}(n,\mathfrak{k})$ of all upper triangular $n\times n$ matrices and $\mathfrak{lt}(n,\mathfrak{k})$ the subalgebra of $\mathfrak{gl}(n,\mathfrak{k})$ of all lower triangular $n\times n$ matrices. Moreover let

$$A=\bigcup_{n\geq 2}\mathfrak{ut}(n,\mathfrak{k}) \text{ and } B=\bigcup_{n\geq 2}\mathfrak{lt}(n,\mathfrak{k}).$$

Then it immediately follows that $L=A+B$. Since

$$S\ni\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&-1\end{pmatrix}\in A+B,$$

$$\text{but }\begin{pmatrix}1&0\\0&0\end{pmatrix}\notin S,\ \begin{pmatrix}0&0\\0&-1\end{pmatrix}\notin S,$$

S is not factorized in $L=A+B$. Furthermore we have

$$S=(A\cap S)+(B\cap S) \text{ but } A\cap B\not\leq S.$$

Therefore we can say that S is a factorized Lie algebra by $(A\cap S)$ and $(B\cap S)$. Proposition 5 implies

$$S<X(S)=A\cap(B+S)+S=B\cap(A+S)+S.$$

Since $A^2\leq S$, $B^2\leq S$, $S\lhd L$, we can confirm by Proposition 6 that $X(S)\lhd L$.

(2) Let $\mathfrak{uto}(n,\mathfrak{k})$ be the subalgebra of $\mathfrak{gl}(n,\mathfrak{k})$ of all upper triangular $n\times n$ matrices (a_{ij}) with $a_{ii}=0$ for any even integer i, and $\mathfrak{lte}(n,\mathfrak{k})$ the subalgebra of $\mathfrak{gl}(n,\mathfrak{k})$ of all lower triangular $n\times n$ matrices (a_{ij}) with $a_{ii}=0$ for any odd integer i. Moreover let

$$A'=\bigcup_{n\geq 2}\mathfrak{uto}(n,\mathfrak{k}) \text{ and } B'=\bigcup_{n\geq 2}\mathfrak{lte}(n,\mathfrak{k}).$$

Then we have $L=A'+B'$ and $A'\cap B'=0$. The same matrices in (1) indicate that S is not factorized in $L=A'+B'$. Thus we obtain

$$S\supsetneqq(A'\cap S)+(B'\cap S)$$

by using Lemma 1.

4. Finiteness Conditions

In this section we shall consider some finiteness conditions of factorized Lie algebras. To do this, we need the following key lemma.

Lemma 7. *Let $L=A+B$ be a Lie algebra factorized by A and B. If H and K are ideals of L such that $H\leq K$, $A\cap H=A\cap K$ and $B\cap(A+H)=B\cap(A+K)$, then $H=K$.*

Proof. By using the modular law, we have

$$\begin{aligned}A+H&=(A+H)\cap(A+B)=A+(B\cap(A+H))\\&=A+(B\cap(A+K))=(A+K)\cap(A+B)=A+K,\end{aligned}$$

and so

$$\begin{aligned}K&=K\cap(A+K)=K\cap(A+H)\\&=(A\cap K)+H=(A\cap H)+H=H.\end{aligned}$$

□

The following theorem is another main result in the paper.

Theorem 8. *Let $L=A+B$ be a Lie algebra factorized by A and B.*

(1) *Let Δ be any of the relations $\leq$, $\lhd$, si, $\lhd^{\alpha}$, asc and assume that $A\Delta L$. If $A\in$ Max-$\lhd$ (resp. Min-$\lhd$) and $B\in$ Max-Δ (resp. Min-Δ), then $L\in$ Max-$\lhd$ (resp. Min-$\lhd$).*

(2) *Assume that A ser $L\in \mathrm{L}\mathfrak{F}$. If $A\in$ Max-$\lhd$ (resp. Min-$\lhd$) and $B\in$ Max-ser (resp. Min-ser), then $L\in$ Max-$\lhd$ (resp. Min-$\lhd$).*

Proof. Let $\{I_n\}_{n\in\mathbb{N}}$ be an ascending chain (resp. a descending chain) of ideals of L. Then $\{A\cap I_n\}_{n\in\mathbb{N}}$ is an ascending chain (resp. a descending chain) of ideals of A.

(1) $\{B\cap(A+I_n)\}_{n\in\mathbb{N}}$ is an ascending chain (resp. a descending chain) of Δ-subalgebras of B. Since $A\in$ Max-$\lhd$ (resp. Min-$\lhd$) and $B\in$ Max-Δ (resp. Min-Δ), there exists a positive integer m such that

$$A \cap I_m = A \cap I_{m+1} = A \cap I_{m+2} = \cdots,$$
$$B \cap (A + I_m) = B \cap (A + I_{m+1}) = B \cap (A + I_{m+2}) = \cdots.$$

It follows from Lemma 7 that $I_m = I_{m+1} = I_{m+2} = \cdots$. Therefore we get $L \in$ Max-$\lhd$ (resp. Min-$\lhd$).

(2) By [1, Proposition 13.2.4] $\{B \cap (A + I_n)\}_{n \in \mathbb{N}}$ is an ascending chain (resp. a descending chain) of serial subalgebras of B. Therefore we obtain that $I_m = I_{m+1} = I_{m+2} = \cdots$ in the same way as (1). □

As a corollary of Theorem 8 we get the following result which corresponds to [2, Lemma 1.2.6].

Corollary 9 . *Let $L = A + B$ be a Lie algebra factorized by A and B. If A and B satisfy the maximal (resp. minimal) condition on subalgebras, then L satisfies the maximal (resp. minimal) condition on ideals.*

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