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Regular Functions with values in a Commutative Subalgebra $\mathbb{C}(\mathbb{C})$ of Matrix Algebra $M(4, \mathbb{R})$[1]

Shingo GOTÔ and Kiyoharu NÔNO

Department of Mathematics, Fukuoka University of Education

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Abstract

In this paper, we construct a commutative algebra $\mathbb{C}(\mathbb{C})$ identified with $\mathbb{C}^2$ as subalgebra of the four dimensional real matrix algebra $M(4, \mathbb{R})$. Next, we give a regularity of functions of two complex variables with values in $\mathbb{C}(\mathbb{C})$ and give some properties of regular functions.

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1. Introduction

In 1934, R. Fueter ([3]) has given a definition of regular functions over the quaternion field $\mathbb{H}$ identified with $\mathbb{R}^4$ by means of extended Cauchy-Riemann equations. A. Sudbery ([11]) developed a quaternionic regular function theory.

In 1971, M. Naser ([6]) gave a regularity (hyperholomorphy) of quaternionic functions using quaternionic differential operator $\frac{\partial}{\partial \overline{z}_1}+e_2\frac{\partial}{\partial \overline{z}_2}$, where e_2 is a base of $\mathbb{H}$ and, $\frac{\partial}{\partial \overline{z}_1}$, $\frac{\partial}{\partial \overline{z}_2}$ are usual complex differential operators. M. Naser and several authors ([6-9]) developed a theory of hyperholomorphic functions as a holomorphic mapping theory on $\mathbb{C}^2$.

Also, R. Delanghe ([2]) gave a regularity of functions with values in Clifford algebra as smooth solusions of generalized Cauchy-Riemann equation and R. Delanghe, F. Brackx and F. Sommen ([1]) have developed a function theory (monogenic function theory). Also, R. Ryan [10] have developed the function theory on complex Clliford Algebra.

In [5], Hurwitz Algebra (Quaternion field and Clifford Algebras) were constructed as non-commutative subalgebra of the matrix alebra.

In this paper, at first, we construct a commutative algebra $\mathbb{C}(\mathbb{C})$ as a commutative subalgebra of the four dimentional real Matrix algebra. In next, we introduce a regularity of functions defined a domain in $\mathbb{C}^2$ with values in $\mathbb{C}(\mathbb{C})$ and give several properties of regular functions.

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[1] The subject of this paper was talked by the authors in the 32th Ouyousuugakukenkyuushuuka (august, 2007)

2. Preliminaries and Definitions

Let $M(4:\mathbb{R})$ be the 4-dimentional Matrix algebra on the field $\mathbb{R}$ of real numbers. Put

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ e_{0i} = \begin{pmatrix} e_i & 0 \\ 0 & e_i \end{pmatrix},\ e_{1i} = \begin{pmatrix} 0 & -e_i \\ e_i & 0 \end{pmatrix} \quad (i = 0,\ 1).$$

Put $\varepsilon_0 = e_{00}$, $\varepsilon_1 = e_{01}$, $\varepsilon_2 = e_{10}$, $\varepsilon_3 = e_{11}$. Then, we have the following relations:

$$\varepsilon_1^2 = -\varepsilon_0,\ \varepsilon_2^2 = -\varepsilon_0,\ \varepsilon_3^2 = \varepsilon_0,$$

$$\varepsilon_1\varepsilon_2 = \varepsilon_3,\ \varepsilon_2\varepsilon_3 = -\varepsilon_1,\ \varepsilon_3\varepsilon_1 = -\varepsilon_2.$$

Then the following algebra $\mathbb{C}(\mathbb{C})$ is a commutative subalgebra of $M(4,\ \mathbb{R})$:

$$\mathbb{C}(\mathbb{C}) = \{z = \varepsilon_0 x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 |\ x_1,\ x_2,\ x_3,\ x_4 \in \mathbb{R}\}.$$

Put $z_0 = \varepsilon_0 x_0 + \varepsilon_1 x_1$, $z_1 = \varepsilon_0 x_2 + \varepsilon_1 x_3$ and $1 = \varepsilon_0$, $\varepsilon = \varepsilon_2$, then, $\mathbb{C}(\mathbb{C})$ is represented by the form:

$$\mathbb{C}(\mathbb{C}) = \{z = z_0 + \varepsilon z_1 |\ z_0,\ z_1 \in \mathbb{R}\}.$$

Then, we identify $\mathbb{C}(\mathbb{C})$ with $\mathbb{C}^2$.

For $z = z_0 + \varepsilon_2 z_1$, $w = w_0 + \varepsilon_2 w_1 \in \mathbb{C}(\mathbb{C})$, the multiplication zw is defined by the following:

$$zw = (z_0 w_0 - z_1 w_1) + \varepsilon_2 (z_0 w_1 + z_1 w_0).$$

Also, the norm $\|z\|$ of $z = z_0 + \varepsilon_2 z_1$ is given by the following matrix norm:

$$\|z\| = \sqrt{tr({}^t zz)}.$$

Next, we consider the following differential operators:

$$D^* = \frac{1}{2}\left(\frac{\partial}{\partial z_0} + \varepsilon \frac{\partial}{\partial z_1}\right),\ D = \frac{1}{2}\left(\frac{\partial}{\partial z_0} - \varepsilon \frac{\partial}{\partial z_1}\right),$$

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2k}} + \varepsilon_1 \frac{\partial}{\partial x_{2k+1}}\right) \quad (k = 0,\ 1).$$

where, $\partial/\partial z_k$ $(k = 0,\ 1)$ are usual complex differential operators. Also, $\partial/\partial x_{2k}$ and $\partial/\partial x_{2k+1}$ are usual real differential operators.

Let G be a domain in $\mathbb{C}^2$. We consider a function f defined in G and with values in $\mathbb{C}(\mathbb{C})$:

$$f = f_0 + \varepsilon_2 f_1 : z = (z_0,\ z_1) \in G \longrightarrow f(z) = f_0(z_0,\ z_1) + \varepsilon_2 f_1(z_0,\ z_1) \in \mathbb{C}(\mathbb{C}),$$

where f_0, f_1 are complex valued functions.

DEFINITION 1. A function $f = f_0 + \varepsilon_2 f_1$ is said to be *S-regular* in G if
(1) f_j $(j = 0,\ 1)$ are holomorphic functions in G,

(2) $D^* f = 0$ in G,
where the differential operator D_1^* operates to f by the following:

$$D^* f = \frac{1}{2}\left\{\left(\frac{\partial}{\partial z_0}+\varepsilon_2\frac{\partial}{\partial z_1}\right)(f_0+\varepsilon_2 f_1)\right\} = \frac{1}{2}\left\{\left(\frac{\partial f_0}{\partial z_0}-\frac{\partial f_1}{\partial z_1}\right)+\varepsilon_2\left(\frac{\partial f_1}{\partial z_0}+\frac{\partial f_0}{\partial z_1}\right)\right\}.$$

Let f be a *S-regular* function defined in G. We define the derivative f' of f by the following:

$$f' = Df.$$

PROPOSITION 1. *Let G be a domain in $\mathbb{C}^2$ and f be a S-regular function defined in G. Then,*

$$f' = \frac{\partial f}{\partial z_0} = \frac{\partial f}{\partial x_0}.$$

Proof. From f is *S-regular* in G, we have

$$\frac{\partial f}{\partial z_0} = -\varepsilon_2\frac{\partial f}{\partial z_1}.$$

Hence,

$$f' = \frac{1}{2}\left(\frac{\partial f}{\partial z_0}-\varepsilon_2\frac{\partial f}{\partial z_1}\right) = \frac{\partial f}{\partial z_0} = \frac{\partial f}{\partial x_0}.$$

3. Properties of holomorphic and H-holomorphic functions

In this section, let G be a domain in $\mathbb{C}^2$. We can obtain the following properties from the definition of *S-regularity*.

PROPOSITION 2.

(1) Let f and g be *S-regular* functions defined in G and $c_1, c_2 \in \mathbb{C}(\mathbb{C})$. Then, $c_1 f + c_2 g$ and fg are also *S-regular* in G. Then, $(c_1 f + c_2 g)' = c_1 f' + c_2 g'$, $(fg)' = f'g + fg'$ in G.

(2) Let f be a *S-regular* function defined in G. Then, the derivative f' is also *S-regular* in G.

(3) Let G_1 and G_2 be domains in $\mathbb{C}^2$ and $f: G_2 \longrightarrow \mathbb{C}(\mathbb{C})$, $g: G_1 \longrightarrow \mathbb{C}(\mathbb{C})$ be functions such that $g(G_1) \subset G_2$. If f and g are *S-regular*, then the composition $(f \circ g)(z) = f(g(z))$ is also *S-regular* in G_1.

(4) $f(z) = z^n$ is *S-regular* in $\mathbb{C}^2$. Then, $f'(z) = nz^{n-1}$ $(n = 1, 2, \ldots)$.

(5) Let Ω be a domain in $\mathbb{C}^2$ and G be a subdomain in Ω. Also, let f and g be *S-regular* functions defined in Ω. If $f = g$ in G, then $f = g$ in Ω.

Put

$$\omega = d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} + \varepsilon_2 dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1}.$$

THEOREM 1. *Let G be a domain in $\mathbb{C}^2$ and D be any domain in G with smooth boundary ∂D such that $\overline{D} \subset G$. If f be a S-regular function in G, then we have that*

$$\int_{\partial D} \omega f = 0.$$

Proof. Becouse of

$$\omega f = (d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} + \varepsilon_2 dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1})(f_0 + \varepsilon_2 f_1)$$
$$= f_0 d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} - f_1 dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1} + \varepsilon_2 (f_1 d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} + f_0 dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1}),$$

we have

$$d(\omega f) = \left(\frac{\partial f_0}{\partial z_0} - \frac{\partial f_1}{\partial z_1}\right) dz_0 \wedge d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} + \varepsilon_2 \left(\frac{\partial f_1}{\partial z_0} + \frac{\partial f_0}{\partial z_1}\right) dz_0 \wedge d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} = 0.$$

in D. By Storkes' Theorem, we obtain the conclusion.

LEMMA *Let f be a homogeneous polynomial of degree m with respect to the variables z_0 and z_1. If f be a S-regular function in $\mathbb{C}^2$, then we have*

$$f(z) = \frac{1}{m!} \frac{\partial^m f(z)}{\partial z_0^m} z^m. \tag{1}$$

Proof. Since $f(z)$ is homogeneous polynomial, then we have

$$f(z) = \frac{1}{m} \frac{\partial f(z)}{\partial z_0} z.$$

From $\dfrac{\partial f(z)}{\partial z_0}$ is a homogeneous polynomial of $m-1$, we have

$$\frac{\partial f(z)}{\partial z_0} = \frac{1}{m-1} \frac{\partial^2 f(z)}{\partial z_0^2} z.$$

Repeating the above argument, we have (1).

TEOREM 2. *Let $f(z)$ be a function defined in a neighbourhood U of $0 \in \mathbb{C}^2$ with values in $\mathbb{C}(\mathbb{C})$. If $f(z)$ has a power series expansion in U:*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then $f(z)$ is S-regular in U.

Proof. From $f(z)$ converges uniformly in U, we have that

$$\left(\frac{\partial}{\partial z_0} + \varepsilon_2 \frac{\partial}{\partial z_1}\right) f(z) = \sum_{n=0}^{\infty} a_n \left(\frac{\partial}{\partial z_0} + \varepsilon_2 \frac{\partial}{\partial z_1}\right) z^n = 0.$$

THEOREM 3. *Let G be a domain in $\mathbb{C}^2$. Let f be a S-regular function in G and $\alpha \in G$. Then, there exists a neighbourhood U of α such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \quad z \in U, \tag{2}$$

where $a_n = \dfrac{1}{n!}\dfrac{\partial^n f(\alpha)}{\partial z_0^n}$ $(n = 0, 1, 2, \cdots)$.

Proof. We may assume without loss of generality that $\alpha = 0$. Since $f(z)$ is a holomorphic function in G with valued in $\mathbb{C}(\mathbb{C})$, there exists a neighbourhood U of 0 such that

$$f(z) = \sum_{n=0}^{\infty} P_n(z),\ z \in U,$$

where $P_n(z)$ are homogenerous polynomials of degree n with respect to the variables z_0 and z_1. Since the series (2) converges uniformly in U, we have

$$D_1^* f(z) = \left(\frac{\partial}{\partial z_0} + \varepsilon_2 \frac{\partial}{\partial z_1}\right) f(z) = \sum_{n=0}^{\infty}\left(\frac{\partial}{\partial z_0} + \varepsilon_2 \frac{\partial}{\partial z_1}\right) P_n(z).$$

From $f(z)$ is S-regular in G and $P_n(z)$ are homogenerous polynomials of degree n with respect to the variables z_0 and z_1, we have that

$$\left(\frac{\partial}{\partial z_0} + \varepsilon_2 \frac{\partial}{\partial z_1}\right) P_n(z) = 0.$$

Hence, $P_n(z)$ is a S-regular function in G. Becouse of

$$\frac{\partial^n f(0)}{\partial z_0^n} = \frac{\partial^n f(z)}{\partial z_0^n},$$

by Proposition 2, we have

$$f(z) = \sum_{n=0}^{\infty} P_n(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n P_n(z)}{\partial z_0^n} z^n = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n f(0)}{\partial z_0^n} z^n = \sum_{n=0}^{\infty} a_n z^n$$

in U.

THEOREM 4. *Let G be a domain of holomorphy in $\mathbb{C}^2$. If $f = f_0 + \varepsilon_2 f_1$ is a S_1-regular function in G, there exists a S-regular function F in G such that $F'(z) = f(z)$.*

Proof. Put

$$\alpha_0 = \overline{f_0}\, d\overline{z_0} - \overline{f_1}\, d\overline{z_1},\ \alpha_1 = \overline{f_1}\, d\overline{z_0} + \overline{f_0}\, d\overline{z_1}.$$

Then, we have

$$\bar\partial \alpha_0 = -\frac{\partial \overline{f_0}}{\partial \overline{z_1}} d\overline{z_0} \wedge d\overline{z_1} - \frac{\partial \overline{f_1}}{\partial \overline{z_0}} d\overline{z_0} \wedge d\overline{z_1} = -\left(\overline{\frac{\partial f_0}{\partial z_1} + \frac{\partial f_1}{\partial z_0}}\right) d\overline{z_0} \wedge d\overline{z_1} = 0,$$

$$\bar\partial \alpha_1 = -\frac{\partial \overline{f_1}}{\partial \overline{z_1}} d\overline{z_0} \wedge d\overline{z_1} + \frac{\partial \overline{f_0}}{\partial \overline{z_0}} d\overline{z_0} \wedge d\overline{z_1} = \left(\overline{\frac{\partial f_0}{\partial z_0} - \frac{\partial f_1}{\partial z_1}}\right) d\overline{z_0} \wedge d\overline{z_1} = 0.$$

From G is domain of holomorphy, there exists a function h_j such that $\bar\partial h_j = \alpha_j$ $(j = 0, 1)$. That is,

$$\frac{\partial h_0}{\partial \overline{z_0}} d\overline{z_0} + \frac{\partial h_0}{\partial \overline{z_1}} d\overline{z_1} = \overline{f_0}\, d\overline{z_0} - \overline{f_1}\, d\overline{z_1},\quad \frac{\partial h_1}{\partial \overline{z_0}} d\overline{z_0} + \frac{\partial h_1}{\partial \overline{z_1}} d\overline{z_1} = \overline{f_1}\, d\overline{z_0} + \overline{f_0}\, d\overline{z_1}.$$

Put

$$F_0 = \overline{h_0},\ F_1 = \overline{h_1}.$$

Then, we have $F'(z) = f(z)$ and

$$\frac{\partial F_0}{\partial z_0} = \frac{\partial F_1}{\partial z_1},\ \frac{\partial F_0}{\partial z_1} = -\frac{\partial F_1}{\partial z_0}.$$

THEOREM 5. *Let G be a domain of holomorphy in $\mathbb{C}^2$. If $f_0(z_0, z_1)$ is a complex valued holomorphic function in G such that $\partial^2 f_0/\partial z_0^2 + \partial^2 f_0/\partial z_1^2 = 0$, there exists a complex valued holomorphic function $f_1(z_0, z_1)$ in G such that $f_0 + \varepsilon_2 f_1$ is S_1-regular in G.*

Proof. Put

$$\alpha = -\frac{\partial \overline{f_0}}{\partial \overline{z_1}} d\overline{z_0} + \frac{\partial \overline{f_0}}{\partial \overline{z_0}} d\overline{z_1}.$$

Since α is $\bar{\partial}$-closed form and G is a domain of holomorphy, there exists a complex valued function g defined in G such that

$$\bar{\partial} g = \alpha.$$

Putting

$$f_1 = \bar{g},$$

we have

$$\frac{\partial f_1}{\partial z_0} = -\frac{\partial f_0}{\partial z_1},\ \frac{\partial f_1}{\partial z_1} = \frac{\partial f_0}{\partial z_0}.$$

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