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Operator balls and completion problems

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Abstract

Studies of the intersection of two operator balls are useful for solving some completion problem of partial operator matrices. In this paper we study the condition which the intersection of two operator balls is trivial or the closed unit ball. Further, we apply to the 2×2 contractive and positive completion problem.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. For selfadjoint operators on $\mathcal{H}$, we shall consider the order is given by positive-deniteness. So an operator $A \in B(\mathcal{H})$ is said to be *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. For selfadjoint operators $A, B \in B(\mathcal{H})$, $A \leq B$ means that $B - A$ is positive. $A \lneq B$ means that $B - A$ is positive and $A \neq B$.

We denote by $\mathcal{B}$ the set of all contractions on $\mathcal{H}$, that is

$$\mathcal{B} = \{K \in B(\mathcal{H}) \ ; \ \|K\| \leq 1\}.$$

Here $\|K\|$ denotes the operator norm of $K \in B(\mathcal{H})$. For two operators $A, B \in B(\mathcal{H})$, we define

$$\mathcal{K}(A, B) = \{AKB \ ; K \in \mathcal{B}\}$$

which will be called a *operator ball*.

For a given partial operator matrix (some entries are specified and some are free), completion problems ask whether there exists a completion satisfying some condition. Contractive completion problems and positive completion problems are studied by many mathematicians. In [5], G. Nævdal and H.J. Woerdeman consider the following completion problem. Let A and B are given selfadjoint contractions on $\mathcal{H}$, that is $A = A^*$, $B = B^*$, $\|A\| \leq 1$, and $\|B\| \leq 1$, then find operator $X \in B(\mathcal{H})$ which satisfies

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq 1 \quad \text{i.e.} \quad -\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Here I denotes the identity operator on $\mathcal{H}$. Using a well-known characterization of 2×2 positive operator matrix (see e.g. [2]), X satisfies

$$\begin{bmatrix} I - A & X \\ X^* & I - B \end{bmatrix} \geq 0$$

if and only if $X \in \mathcal{K}((I - A)^{1/2}, (I - B)^{1/2})$. Hence the solution set of this completion problem is the intersection of two operator balls $K((I - A)^{1/2}, (I - B)^{1/2})$ and $K((I + A)^{1/2}, (I + B)^{1/2})$. The authors study the intersection of two operator balls and above completion problems in the finite dimensional case.

In this paper, we shall consider an alternative approach to studies of the intersection of two operator balls and some completion problems.

2. Preliminaries

The structure of operator balls is studied by Ju.L. Smul'jan (see e.g. [6]). First, let us recall the inclusion of operator balls.

Theorem 1 *Let A, B, C, and D be non-zero operators on $\mathcal{H}$. Then $\mathcal{K}(A, B) \subset \mathcal{K}(C, D)$ holds if and only if there exists a positive number ρ such that*

$$AA^* \leq \rho CC^*, \quad B^*B \leq \rho^{-1} D^*D.$$

Hence $\mathcal{K}(A, B) = \mathcal{K}(C, D)$ holds if and only if there exists a positive number ρ such that

$$AA^* = \rho CC^*, \quad B^*B = \rho^{-1} D^*D.$$

For an operator ball $\mathcal{K}(A, B)$, the operators A and B are called its left and right radii, respectively. By Theorem 1, for any operator A and B, we have

$$\mathcal{K}(A, B) = \mathcal{K}(|A^*|, |B|)$$

where $|A^*| = (AA^*)^{1/2}$ and $|B| = (B^*B)^{1/2}$. So we may assume that left and right radii of operator balls are always positive.

In Section 3, we shall need the parallel sum of two positive operators. Let A and B be positive operators on $\mathcal{H}$. Since $A \leq A + B$ and $B \leq A + B$, by a result of R.G. Douglas (see e.g. [3]) there are uniquely determined operators $C, D \in B(\mathcal{H})$ such that

$$A^{1/2} = (A + B)^{1/2}C, \quad \ker(A + B)^{1/2} \subset \ker C^*$$

and

$$B^{1/2} = (A + B)^{1/2}D, \quad \ker(A + B)^{1/2} \subset \ker D^*.$$

Here ker T denotes the kernel of an operator $T \in B(\mathcal{H})$. We define the operator

$$A : B = A^{1/2}C^*DB^{1/2}$$

which will be called the *parallel sum* of A and B.

The parallel sum $A : B$ is a positive operator on $\mathcal{H}$ and satisfies the following properties.

(i) $A : B = B : A$.

(ii) $A : B \leq$ A and $A : B \leq B$.

(iii) If $A + B$ is invertible, then $A : B = A(A + B)^{-1}B = B(A + B)^{-1}A$.

(iv) $\operatorname{ran}(A : B)^{1/2} = \operatorname{ran} A^{1/2} \cap \operatorname{ran} B^{1/2}$.

Here ran T denotes the range of an operator $T \in B(\mathcal{H})$.

For a further discussion of the parallel sum, we refer the reader to [1] and [4].

3. Intersection of two operator balls and completion problem

In this section, we shall consider the condition which the intersection of two operator balls is trivial or the closed unit ball. Further, we apply the result to the 2×2 contractive and positive completion problem where the diagonal entries are specified. First we consider the trivial case.

Theorem 2 *Let A, B, C, and D be positive operators on $\mathcal{H}$. The intersection of two operator balls $\mathcal{K}(A, B)$ and $\mathcal{K}(C, D)$ is trivial, if and only if either $A^2 : C^2 = 0$ or $B^2 : D^2 = 0$.*

Proof. We shall prove that $\mathcal{K}(A, B) \cap \mathcal{K}(C, D) \neq \{0\}$ if and only if $A^2 : C^2 \neq 0$ and $B^2 : D^2 \neq 0$.

Suppose that $\mathcal{K}(A, B) \cap \mathcal{K}(C, D) \neq \{0\}$, then there is a non-zero operator X which belongs to $\mathcal{K}(A, B) \cap \mathcal{K}(C, D)$. Hence there exist contractions $K, L \in B(\mathcal{H})$ such that

$$X = AKB = CLD.$$

Since X is a non-zero operator,

$$\operatorname{ran}(A^2 : C^2)^{1/2} = \operatorname{ran} A \cap \operatorname{ran} C \neq \{0\}$$

and hence we obtain $A^2 : C^2 \neq 0$. Similarly, from $BK^*A = DL^*C \neq 0$, we obtain $B^2 : D^2 \neq 0$.

Conversely suppose that $A^2 : B^2 \neq 0$ and $C^2 : D^2 \neq 0$. Then both $(A^2 : C^2)^{1/2}$ and $(B^2 : D^2)^{1/2}$ are non-zero operators, and hence there exists a contraction K such that

$$(A^2 : C^2)^{1/2} K (B^2 : D^2)^{1/2} \neq 0.$$

Since $A^2 : C^2 \leq A^2$ and $B^2 : D^2 \leq B^2$, by virtue of Theorem 1 we obtain

$$\mathcal{K}((A^2 : C^2)^{1/2}, (B^2 : D^2)^{1/2}) \subset \mathcal{K}(A, \mathrm{B}).$$

Similarly we obtain $\mathcal{K}((A^2 : C^2)^{1/2}, (B^2 : D^2)^{1/2}) \subset \mathcal{K}(C, D)$ and hence

$$\mathcal{K}((A^2 : C^2)^{1/2}, (B^2 : D^2)^{1/2}) \subset \mathcal{K}(A, B) \cap \mathcal{K}(C, D).$$

This proves $\mathcal{K}(A, B) \cap \mathcal{K}(C, D) \neq \{0\}$. $\square$

By the proof of Theorem 2, if $\mathcal{K}(A, B) \cap \mathcal{K}(C, D) \neq \{0\}$, then $\mathcal{K}(A, B) \cap \mathcal{K}(C, D)$ contains a non-trivial operator ball $\mathcal{K}((A^2 : C^2)^{1/2}, (B^2 : D^2)^{1/2})$.

We apply Theorem 2 to the some completion problem. For given contractive and positive operators $A, B \in B(\mathcal{H})$, we consider the contractive and positive completion problem of the partial operator matrix

$$\begin{bmatrix} A & ? \\ ? & B \end{bmatrix} \tag{1}$$

where ? means unspecified entries. Obviously, zero operator is a solution of this completion problem, so we consider the trivial case.

Corollary 3 *Let A, B be contractive and positive operators on $\mathcal{H}$. The solution set of the completion problem (1) is trivial, that is*

$$\left\{X \in B(\mathcal{H});\, 0 \le \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \le \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right\} = \{0\}$$

if and only if either of A or B is an orthogonal projection.

Proof. For an operator $X \in B(\mathcal{H})$, the operator matrix

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \tag{2}$$

is positive if and only if $X \in \mathcal{K}(A^{1/2}, B^{1/2})$. The operator matrix (2) is contractive if and only if $X \in \mathcal{K}((I-A)^{1/2}, (I-B)^{1/2})$. So we have

$$\left\{X;\, 0 \le \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \le \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right\} = \mathcal{K}(A^{1/2}, B^{1/2}) \cap \mathcal{K}((I-A)^{1/2}, (I-B)^{1/2}).$$

From Theorem 2, the intersection of $\mathcal{K}(A^{1/2}, \mathrm{B}^{1/2})$ and $\mathcal{K}((I-A)^{1/2}, (I-B)^{1/2})$ is trivial if and only if either $A : (I-A) = 0$ or $B : (I-B) = 0$. Since $A : (I-A) = A - A^2$, this completes the proof. □

For given two operator balls, if one ball is the closed unit ball $\mathcal{B}$ and the other ball contains $\mathcal{B}$ then the intersection of these operator balls is $\mathcal{B}$. We shall prove that the converse is true.

Theorem 4 *Let A, B, C, and D be positive operators on $\mathcal{H}$. If the intersection of two operator balls $\mathcal{K}(A, B)$ and $\mathcal{K}(C, D)$ is the closed unit ball $\mathcal{B}$, then either $\mathcal{K}(A, B) = \mathcal{B}$ or $\mathcal{K}(C, D) = \mathcal{B}$.*

Proof. Suppose that the intersection of $\mathcal{K}(A, B)$ and $\mathcal{K}(C, D)$ is the closed unit ball $\mathcal{B}$, that is

$$\mathcal{K}(A, B) \cap \mathcal{K}(C, D) = \mathcal{B}.$$

Since both $\mathcal{K}(A, B)$ and $\mathcal{K}(C, D)$ contain $\mathcal{B}$, we may assume that $A^2 \ge I$, $B^2 \ge I$, $C^2 \ge I$, and $D^2 \ge \mathrm{I}$. We shall show that if $\mathcal{K}(A, B) \ne \mathcal{B}$ and $\mathcal{K}(C, D) \ne \mathcal{B}$ then there is a contradiction to $\mathcal{K}(A, B) \cap \mathcal{K}(C, D) = \mathcal{B}$.

We assume that $\mathcal{K}(A, B) \ne \mathcal{B}$ and $\mathcal{K}(C, D) \ne \mathcal{B}$. Then $\mathcal{K}(A, B) \supsetneq \mathcal{B}$ implies

$$A^2 \gneq I \text{ or } B^2 \gneq I$$

and similarly $\mathcal{K}(C, D) \supsetneq \mathcal{B}$ implies

$$C^2 \gneq I \text{ or } D^2 \gneq I.$$

So we may consider following two cases.

(Case 1) We shall consider the case $A^2 \gneq I$ and $C^2 \gneq I$. By assumption, there exist $\alpha \in (0, 1)$ and rank one projections $P, Q \in \mathcal{B}(\mathcal{H})$ such that

$$I+\alpha P \le A^2, \quad I+\frac{\alpha}{1-\alpha}Q \le C^2.$$

Since $0 < \alpha < 1$, we obtain

$$I+\alpha Q \le I+\frac{\alpha}{1-\alpha}Q \le C^2.$$

(i) If $P = Q$, then $I + \alpha P \le A^2$ and $I + \alpha P \le C^2$, and hence we have

$$(I+\alpha P)^{1/2} \in \mathcal{K}(A, I) \cap \mathcal{K}(C, I) \subset \mathcal{K}(A, B) \cap \mathcal{K}(C, D).$$

But $(I+\alpha P)^{1/2}$ is not a contraction, so this is a contradiction.

(ii) If $PQ = 0$, then ran P is orthogonal to ran$Q$ and $P + Q$ is an orthogonal projection. So we consider matrix representations on ran $P \oplus$ ran$Q$, then

$$P=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad Q=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Put

$$\beta=\left\|(I+\alpha P)^{-1/2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\|=\left\|(I+\alpha Q)^{-1/2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\|=\sqrt{2\{(1+\alpha)^{-1}+1\}}$$

and

$$K_1=\frac{1}{\beta}(I+\alpha P)^{-1/2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \quad K_2=\frac{1}{\beta}(I+\alpha Q)^{-1/2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},$$

then K_1 and K_2 are contractions. We consider the operator

$$X=\frac{1}{\beta}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

then

$$X=(I+\alpha P)^{1/2}K_1=(I+\alpha Q)^{1/2}K_2 \in \mathcal{K}((I+\alpha P)^{1/2}, I) \cap \mathcal{K}((I+\alpha Q)^{1/2}, I)$$

and hence $X \in \mathcal{K}(A, B) \cap \mathcal{K}(C, D)$. However, since $\beta < 2$, X is not a contraction. So this is a contradiction.

(iii) We shall suppose that $P \ne Q$ and $PQ \ne 0$. Consider the polar decomposition

$$(I-\alpha Q)^{1/2}(I+\alpha P)^{1/2} = VR \tag{3}$$

where V is a unitary operator on $\mathcal{H}$ and $R = |(I-\alpha Q)^{1/2}(I+\alpha P)^{1/2}|$.

If we assume that $R \le I$, then $R^2 \le I$ and hence

$$(I+\alpha P)^{1/2}(I-\alpha Q)(I+\alpha P)^{1/2} \leq I \quad \text{i.e.} \quad I-\alpha Q \leq (I+\alpha P)^{-1}.$$

Since $(I-\alpha Q)^{-1} = I+\alpha(1-\alpha)^{-1}Q$, we obtain

$$I+\alpha P \leq (I-\alpha Q)^{-1} = I+\frac{\alpha}{1-\alpha}Q \quad \text{i.e.} \quad P \leq \frac{1}{1-\alpha}Q.$$

But this contradicts $P \neq Q$, so $R \nleq I$. The similar argument provides $R \ngeq I$.

We shall consider matrix representations on the subspace $\mathcal{M}$ which generated by ran P and ran Q. Since $R \nleq I$ and $R \ngeq I$, there exist $\lambda, \mu \in (0,1)$ and unitary operator U on $\mathcal{M}$ such that

$$R = U\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \mu \end{bmatrix} U^*. \tag{4}$$

By computing the trace of $R^2 = (I+\alpha P)^{1/2}(I-\alpha Q)(I+\alpha P)^{1/2}$;

$$\mathrm{tr}R^2 = \mathrm{tr}\ (I+\alpha P)(I-\alpha Q) = \mathrm{tr}\ I - \alpha^2\ \mathrm{tr}\ PQ < 2,$$

so we obtain $\lambda^{-2}+\mu^2 < 2$. By using (3) and (4) we can define

$$X = (I+\alpha P)^{1/2}\, U\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} = (I-\alpha Q)^{-1/2}\, VU\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}.$$

Since $(I-\alpha Q)^{-1} = I+\alpha(1-\alpha)^{-1}Q$, we obtain

$$X = \left(I+\frac{\alpha}{1-\alpha}Q\right)^{1/2} VU\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

and hence

$$X \in \mathcal{K}((I+\alpha P)^{1/2}, I) \cap \mathcal{K}((I+\alpha(1-\alpha)^{-1}Q)^{1/2}, I) \subset \mathcal{K}(A,B) \cap \mathcal{K}(C,D).$$

But X is not a contraction, so this is a contradiction. In fact, if X is a contraction, then $X^*X \leq I$ and hence

$$\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} U^*(I+\alpha P)U\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} U^*V^*(I-\alpha Q)^{-1}VU\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \leq I.$$

Put $\widetilde{P} = U^*PU$ and $\widetilde{Q} = U^*V^*QVU$, then

$$I+\alpha\widetilde{P} \leq \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \lambda^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(I-\alpha\widetilde{Q})^{-1} \leq \begin{bmatrix} 1 & 0 \\ 0 & \mu^{-2} \end{bmatrix}, \quad \text{i.e.} \quad I-\alpha\widetilde{Q} \geq \begin{bmatrix} 1 & 0 \\ 0 & \mu^2 \end{bmatrix}.$$

Since $\widetilde{P}$ and $\widetilde{Q}$ are rank one projections, we can obtain that

$$\widetilde{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} (I + \alpha\widetilde{P}) \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} (I - \alpha\widetilde{Q})^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix},$$

which implies

$$\begin{bmatrix} \lambda^2(1+\alpha) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mu^2(1-\alpha)^{-1} \end{bmatrix}$$

and hence $\lambda^2 = (1+\alpha)^{-1}$ and $\mu^2 = 1-\alpha$. This contradicts $\lambda^{-2} + \mu^2 < 2$, so X is not a contraction.

In the case of $B^2 \gneq I$ and $D^2 \gneq I$, there is a contradiction similarly.

(Case 2) We shall consider the case $A^2 \gneq I$ and $D^2 \gneq I$. By assumption, there exist $\alpha \in (0, 1)$ and rank one projections P, Q such that

$$I + \alpha P \leq A^2, \quad I + \alpha Q \leq D^2.$$

Since $(I + \alpha P)^{1/2}$ is unitarily similar to $(I + \alpha Q)^{1/2}$, there exists a unitary operator U such that

$$U^*(I + \alpha P)^{1/2} U = (I + \alpha Q)^{1/2}.$$

Put $X = (I + \alpha P)^{1/2} U = U(I + \alpha Q)^{1/2}$, then

$$X \in \mathcal{K}((I + \alpha P)^{1/2}, I) \cap \mathcal{K}(I, (I + \alpha Q)^{1/2})$$

and hence $X \in \mathcal{K}(A, B) \cap \mathcal{K}(C, D)$. But X is not a contraction, so this is a contradiction.

In the case of $B^2 \gneq I$ and $C^2 \gneq I$, there is a contradiction similarly. □

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