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VALUATIONS OF LIE ALGEBRAS

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Abstract. We introduce the concept of valuations of Lie algebras and completions of valued Lie algebras. We show that if $(L, |\cdot|)$ is a valued Lie algebra then there is the unique completion $(L^*, |\cdot|^*)$ up to isomorphism. We show that if a subalgebra H of L is a weak subideal of L (resp. a subideal of L, finite-dimensional, soluble, nilpotent) then its topological closure H^* in $(L^*, |\cdot|^*)$ is a weak subideal of L^* (resp. a subideal of L^*, finite-dimensional, soluble, nilpotent). We also introduce valuations $|\cdot|$ of the generalized Witt algebra $W_{\mathbb{Z}}$ and present some properties of $(W_{\mathbb{Z}}^*, |\cdot|)$. Finally we illustrate the completion of $(W_{\mathbb{Z}}, |\cdot|)$ with a specific example.

Introduction

The study of valuations of fields was initiated in connection with the arithmetic of number fields. In this paper we shall introduce the concept of valuations of Lie algebras, and refer to the pair $(L, |\cdot|)$ of a Lie algebra L and a valuation $|\cdot|$ of L as a valued Lie algebra. The purpose of this paper is first to investigate elementary properties of valued Lie algebras, and secondly to investigate further properties of valuations of generalized Witt algebras over any field of characteristic zero.

In Section 2 we shall introduce the concept of valued Lie algebras and their completions, and prove that every valued Lie algebra has the unique completion up to isomorphism (Theorem 5).

In Section 3 we shall prove that if H is an n-step weak subideal (resp. subideal) of a Lie algebra L then the topological closure H^* in the complete metric space $(L^*, |\cdot|)$ is also an n-step weak subideal (resp. subideal) of L^*, and that if H is a finite-dimensional (resp. soluble, nilpotent) subalgebra of L then H^* is also finite-dimensional (resp. soluble, nilpotent) (Theorem 10).

In Section 4 we shall first introduce valuations $|\cdot|$ of generalized Witt algebras W_G over any field $\mathfrak{k}$ of characteristic zero, where G is a non-trivial subgroup of the additive group $\mathfrak{k} \cap \mathbb{R}$, and secondly prove that every non-zero weak subideal of $W_{\mathbb{Z}}$ is a dense subset of the complete metric space $(W_{\mathbb{Z}}^*, |\cdot|)$ (Theorem 12).

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In Section 5 we shall construct a new Lie algebra $W_{-\infty}$ containing the generalized Witt algebra $W_{\mathbb{Z}}$ over a field $\mathfrak{k}$ of characteristic zero, and prove that $(W_{-\infty}, |\cdot|)$ is complete and $W_{\mathbb{Z}}$ is a dense subset of the metric space $(W_{-\infty}, |\cdot|)$ (Theorem 15).

1. Notation and terminology

Throughout the paper we are always concerned with Lie algebras which are not necessarily finite-dimensional over an arbitrary field $\mathfrak{k}$ unless otherwise specified. Notation and terminology is mainly based on [1]. In this section we explain some symbols and terms which we use here.

The symbol $\mathbb{Z}$ (resp. $\mathbb{R}$) denotes the ring of integers (resp. the field of real numbers). For a field $\mathfrak{k}$, the symbol $\mathfrak{k}^{\times}$ denotes the set of all non-zero elements of $\mathfrak{k}$, that is, $\mathfrak{k}^{\times}$ means the multiplicative group of $\mathfrak{k}$.

Let L be a Lie algebra over $\mathfrak{k}$ and n a non-negative integer. The symbol $H \leq L$ (resp.$H \lhd L$) denotes that H is a subalgebra (resp. an ideal) of L. Angular brackets $\langle\ \rangle$ denotes the subalgebra generated by their contents. In [3] a subalgebra H of L is said to be an n-step weak subideal of L, which is denoted by $H \leq^n L$, provided there exists an ascending chain $(H_i)_{i \leqq n}$ of subspaces of L such that

(a) $H_0 = H$ and $H_n = L$,

(b) $[H_{i+1}, H] \subseteq H_i$ $(0 \leqq i < n)$.

Then the chain $(H_i)_{i \leqq n}$ is said to be a weak series from H to L. In particular, $(H_i)_{i \leqq n}$ is a series from H to L and H is an n-step subideal of L, denoted by $H \lhd^n L$, provided $H_i \lhd H_{i+1} \leq L$ $(0 \leqq i < n)$. H is said to be a weak subideal (resp. a subideal) of L, denoted by H wsi L (resp. H si L), if $H \leq^n L$ (resp. $H \lhd^n L$) for some integer $n \geqq 1$. Let ω be the least limit ordinal. Furthermore, in [3] a subalgebra H of L is said to be an ω-step weakly ascendant subalgebra of L, which is denoted by $H \leq^{\omega} L$, provided there exists an ascending chain $(H_i)_{i \leqq \omega}$ of subspaces of L such that

(a) $H_0 = H$ and $H_{\omega} = L$,

(b) $[H_{i+1}, H] \subseteq H_i$ $(0 \leqq i < \omega)$,

(c) $H_{\omega} = \cup_{i<\omega} H_i$.

A class $\mathfrak{X}$ is a collection of Lie algebras together with their isomorphic copies and 0-dimensional Lie algebras. The symbol $\mathfrak{F}$ (resp. $\mathfrak{A}^n, \mathrm{E}\mathfrak{A}, \mathfrak{N}_n, \mathfrak{N}$) denotes the class of Lie algebras which are finite-dimensional (resp. soluble of derived length $\leqq n$, soluble, nilpotent of class $\leqq n$, nilpotent).

2. VALUED LIE ALGEBRAS AND THEIR COMPLETIONS

In this section we shall introduce the concept of valued Lie algebras and their completions.

Let L be a Lie algebra over $\mathfrak{k}$. We say that a mapping $x \mapsto |x|$ from L to $\mathbb{R}$ is a (non-Archimedian) valuation of L if the mapping satisfies the following conditions (i) – (iv), where $x, y \in L$ and $\alpha \in \mathfrak{k}^{\times}$:

(i) $|x| > 0$ except that $|0| = 0$.

(ii) $|\alpha x| = |x|$.

(iii) $|x + y| \leqq \max\{|x|, |y|\}$.

(iv) $|[x, y]| \leqq |x||y|$.

Then we simply denote by $|\cdot|$ the mapping $x \mapsto |x|$ and say that the pair $(L, |\cdot|)$ is a valued Lie algebra.

A valuation $|\cdot|$ of L is non-trivial if $|x| \neq |y|$ for some non-zero elements x, y of L. Every Lie algebra has a trivial valuation. In fact, let $c \in \mathbb{R}$ and $c \geqq 1$. Then we can easily define a trivial valuation $|\cdot|$ of L by $|x| = c$ $(x \neq 0)$ and $|0| = 0$.

Let $(L, |\cdot|)$ be a valued Lie algebra. We can define a metric function δ on L by $\delta(x, y) = |x - y|$ $(x, y \in L)$. Regard $(L, |\cdot|)$ as a metric space with respect to δ. Let $\{x_n\}_{n \geqq 1}$ be a sequence of elements of L and let $x \in L$. As usual, we say that $\{x_n\}_{n \geqq 1}$ is a Cauchy sequence in $(L, |\cdot|)$ if $\lim_{m,n\to\infty} |x_m - x_n| = 0$, and that $\{x_n\}_{n \geqq 1}$ is a convergent sequence in $(L, |\cdot|)$ with limit x, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} |x_n - x| = 0$. Furthermore, we say that $(L, |\cdot|)$ is a complete valued Lie algebra if every Cauchy sequence in $(L, |\cdot|)$ is a convergent sequence. That is, a valued Lie algebra $(L, |\cdot|)$ is complete if $(L, |\cdot|)$ is a complete metric space.

Example 1. Let X be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{x_i : i = 0, 1, \dots\}$ and σ a derivation of X such that $x_0\sigma = 0$ and $x_i\sigma = x_{i-1}$ $(i \geqq 1)$. Form the split extension $L = X \dotplus \langle\sigma\rangle$ of X by $\langle\sigma\rangle$. For any element

$$x = \lambda_n x_n + \cdots + \lambda_0 x_0 + \lambda_{-1}\sigma$$

of L (where $n \geqq 0$ and $\lambda_i \in \mathfrak{k}$ $(-1 \leqq i \leqq n)$), we define

$$\max(x) = \begin{cases} n & \text{if } \ \lambda_n \neq 0, \\ -1 & \text{if } \ \lambda_i = 0 \ \ (0 \leqq i \leqq n) \text{ and } \lambda_{-1} \neq 0, \\ -\infty & \text{if } \ \lambda_i = 0 \ \ (-1 \leqq i \leqq n). \end{cases}$$

Then it is not hard to see that the mapping $x \mapsto \max(x)$ satisfies the following conditions (i) – (iii), where $x, y \in L$ and $\alpha \in \mathfrak{k}^\times$:

(i) $\max(\alpha x) = \max(x)$.
(ii) $\max(x + y) \leqq \max\{\max(x), \max(y)\}$.
(iii) $\max([x, y]) \leqq \max(x) + \max(y)$.

Hence we can define a non-trivial valuation $|\cdot|$ of L by $|x| = 2^{\max(x)}$ $(x \in L)$. Then for any non-zero element x of L we have $|x| \geqq 2^{-1}$. Thus $(L, |\cdot|)$ is a complete valued Lie algebra.

We begin with the following lemma, which presents a property of convergent sequences of a valued Lie algebra.

Lemma 2. *Let F be the free Lie algebra on a countably infinite set $\{t_1, t_2, \dots\}$ and $w = w(t_1, \dots, t_r)$ a word of F in variables $t_1, \dots, t_r$ (cf. [1, p. 274]). Let $(L, |\cdot|)$ be a valued Lie algebra and $\{x_{i,n}\}_{n \geqq 1}$ a convergent sequence in $(L, |\cdot|)$ with limit $x_i \in L$ $(1 \leqq i \leqq r)$. Then $\{w(x_{1,n}, \dots, x_{r,n})\}_{n \geqq 1}$ is a convergent sequence in $(L, |\cdot|)$ with limit $w(x_1, \dots, x_r)$.*

Proof. Clearly we may assume that w is a monomial word of type $[t_1, \dots, t_s]$. We use induction on s to show that

$$\lim_{n\to\infty} [x_{1,n}, \dots, x_{s,n}] = [x_1, \dots, x_s].$$

It is trivial for $s = 1$. Let $s \geqq 1$ and suppose that the result is true for s. There exists a real number $M > 0$ and an integer $n_0 \geqq 1$ such that $|x_{s+1,n}| < M$ for all $n \geqq n_0$. Then for any $n \geqq n_0$ we have

$$
\begin{aligned}
&|[x_{1,n},\ldots,x_{s,n},x_{s+1,n}]-[x_1,\ldots,x_s,x_{s+1}]| \\
&\quad \leqq \max\{|[[x_{1,n},\ldots,x_{s,n}]-[x_1,\ldots,x_s],x_{s+1,n}]|,|[[x_1,\ldots,x_s],x_{s+1,n}-x_{s+1}]|\} \\
&\quad \leqq \max\{|[x_{1,n},\ldots,x_{s,n}]-[x_1,\ldots,x_s]||x_{s+1,n}|,|[x_1,\ldots,x_s]||x_{s+1,n}-x_{s+1}|\} \\
&\quad \leqq \max\{M|[x_{1,n},\ldots,x_{s,n}]-[x_1,\ldots,x_s]|,|[x_1,\ldots,x_s]||x_{s+1,n}-x_{s+1}|\}.
\end{aligned}
$$

Therefore $\lim_{n\to\infty}[x_{1,n},\ldots,x_{s,n},x_{s+1,n}] = [x_1,\ldots,x_s,x_{s+1}]$. This completes the proof. □

Let $(L,|\cdot|)$ be a valued Lie algebra over $\mathfrak{k}$. We define the completion of $(L,|\cdot|)$ to be a complete valued Lie algebra $(L^*,|\cdot|^*)$ satisfying the following conditions (i) – (iii):

(i) $L \leq L^*$.

(ii) $|x|^* = |x|$ for all $x \in L$.

(iii) L is a dense subset of the metric space $(L^*,|\cdot|^*)$.

Let $(L_1,|\cdot|_1)$ and $(L_2,|\cdot|_2)$ be valued Lie algebras over $\mathfrak{k}$. A Lie isomorphism θ from L_1 onto L_2 is an isomorphism from $(L_1,|\cdot|_1)$ onto $(L_2,|\cdot|_2)$ if θ is an isometric mapping from the metric space $(L_1,|\cdot|_1)$ to the metric space $(L_2,|\cdot|_2)$ (i.e. $|x|_1 = |\theta(x)|_2$ for all $x \in L_1$). Then we usually say that $(L_1,|\cdot|_1)$ is isomorphic to $(L_2,|\cdot|_2)$, denoted by $(L_1,|\cdot|_1) \cong (L_2,|\cdot|_2)$.

Lemma 3. *Let $(L^*,|\cdot|^*)$ be the completion of a valued Lie algebra $(L,|\cdot|)$ over $\mathfrak{k}$ and let $(L^\sim,|\cdot|^\sim)$ be a complete valued Lie algebra over $\mathfrak{k}$. Assume that there exists a Lie monomorphism θ from L to $L^\sim$ such that $|x| = |\theta(x)|^\sim$ for all $x \in L$. Then there exists one and only one Lie monomorphism θ^* from L^* to $L^\sim$ such that $\theta^*|_L = \theta$ and $|x^*|^* = |\theta^*(x^*)|^\sim$ for all $x^* \in L^*$.*

Proof. Let $x^* \in L^*$. Since L is a dense subset of the metric space $(L^*,|\cdot|^*)$, there exists a convergent sequence $\{x_n\}_{n\geqq 1}$ in $(L^*,|\cdot|^*)$ with limit x^* such that $x_n \in L$ for all $n \geqq 1$. Then $\{\theta(x_n)\}_{n\geqq 1}$ is a Cauchy sequence in $(L^\sim,|\cdot|^\sim)$. Since $(L^\sim,|\cdot|^\sim)$ is complete, there exists one and only one element $x^\sim$ of $L^\sim$ such that $\lim_{n\to\infty}\theta(x_n) = x^\sim$. Then we can define a map θ^* from L^* to $L^\sim$ by $\theta^*(x^*) = x^\sim$. In fact, let $\{x'_n\}_{n\geqq 1}$ be another convergent sequence in $(L^*,|\cdot|^*)$ with limit x^* such that $x'_n \in L$ for all $n \geqq 1$. Then for all $n \geqq 1$

$$
\begin{aligned}
|\theta(x'_n) - x^\sim|^\sim &\leqq \max\{|x'_n - x_n|, |\theta(x_n) - x^\sim|^\sim\} \\
&\leqq \max\{|x'_n - x^*|^*, |x^* - x_n|^*, |\theta(x_n) - x^\sim|^\sim\}.
\end{aligned}
$$

Hence we have $\lim_{n\to\infty}\theta(x'_n) = x^\sim$.

Let $y^* \in L^*$ and let $\{y_n\}_{n\geqq 1}$ be a convergent sequence in $(L^*,|\cdot|^*)$ with limit y^* such that $y_n \in L$ for all $n \geqq 1$. Let $\alpha,\beta \in \mathfrak{k}$. Then by Lemma 2 we have $\lim_{n\to\infty}(\alpha x_n + \beta y_n) = \alpha x^* + \beta y^*$ and $\lim_{n\to\infty}[x_n,y_n] = [x^*,y^*]$. Hence

$$
\begin{aligned}
\theta^*(\alpha x^* + \beta y^*) &= \lim_{n\to\infty}\theta(\alpha x_n + \beta y_n) \\
&= \lim_{n\to\infty}\{\alpha\theta(x_n) + \beta\theta(y_n)\} \\
&= \alpha\theta^*(x^*) + \beta\theta^*(y^*)
\end{aligned}
$$

and

$$\begin{aligned}\theta^*([x^*, y^*]) &= \lim_{n\to\infty} \theta([x_n, y_n]) \\ &= \lim_{n\to\infty} [\theta(x_n), \theta(y_n)] \\ &= [\theta^*(x^*), \theta^*(y^*)].\end{aligned}$$

Therefore θ^* is a Lie homomorphism. Moreover, we have

$$|x^*|^* = \lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |\theta(x_n)|^{\sim} = |\theta^*(x^*)|^{\sim}.$$

It follows that θ^* is a Lie monomorphism from L^* to $L^{\sim}$. Clearly we have $\theta^*|_L = \theta$.

Finally, suppose that there exists another Lie monomorphism θ' from L^* to $L^{\sim}$ such that $\theta'|_L = \theta$ and $|x^*|^* = |\theta'(x^*)|^{\sim}$ for all $x^* \in L^*$. Then for all $n \geqq 1$

$$\begin{aligned}|\theta^*(x^*) - \theta'(x^*)|^{\sim} &\leqq \max\{|\theta^*(x^*) - \theta^*(x_n)|^{\sim}, |\theta'(x_n) - \theta'(x^*)|^{\sim}\} \\ &= |x_n - x^*|^*.\end{aligned}$$

Thus we obtain $\theta^* = \theta'$. □

Next let us construct the completion of a valued Lie algebra $(L, |\cdot|)$ over $\mathfrak{k}$. We denote by $\mathrm{CS}(L)$ the collection of all Cauchy sequences in $(L, |\cdot|)$. Let $\{x_n\}_{n\geqq 1}, \{y_n\}_{n\geqq 1} \in \mathrm{CS}(L)$ and let $\{x_n\}_{n\geqq 1} \sim \{y_n\}_{n\geqq 1}$ mean that $\lim_{n\to\infty} |x_n - y_n| = 0$. Then the relation $\sim$ is an equivalence relation on $\mathrm{CS}(L)$. Let $\{x_n\}^*_{n\geqq 1}$ denote the equivalence class represented by $\{x_n\}_{n\geqq 1} \in \mathrm{CS}(L)$ and let L^* denote the collection of all $\sim$-equivalence classes. Let $\alpha \in \mathfrak{k}$. As in the proof of Lemma 2, it is easy to show that $\{x_n + y_n\}_{n\geqq 1}, \{\alpha x_n\}_{n\geqq 1}, \{[x_n, y_n]\}_{n\geqq 1} \in \mathrm{CS}(L)$. Therefore we can define an addition, a scalar multiplication and a bracket product in L^* as follows:

$$\begin{aligned}\{x_n\}^*_{n\geqq 1} + \{y_n\}^*_{n\geqq 1} &= \{x_n + y_n\}^*_{n\geqq 1}, \\ \alpha\{x_n\}^*_{n\geqq 1} &= \{\alpha x_n\}^*_{n\geqq 1}, \\ [\{x_n\}^*_{n\geqq 1}, \{y_n\}^*_{n\geqq 1}] &= \{[x_n, y_n]\}^*_{n\geqq 1}.\end{aligned}$$

It is not hard to see that these operations are well defined and make L^* a Lie algebra over $\mathfrak{k}$. Furthermore, the original Lie algebra L is naturally imbedded in L^* by identifying an element x of L with the equivalence class $\{x\}^*$ represented by the constant sequence $\{x\}_{n\geqq 1}$. Hence we regard L as a subalgebra of L^*.

Let $\{x_n\}^*_{n\geqq 1} \in L^*$. Since $\{|x_n|\}_{n\geqq 1}$ is a convergent sequence in $\mathbb{R}$, a real-valued function $|\{x_n\}^*_{n\geqq 1}|$ on L^* is well defined by $|\{x_n\}^*_{n\geqq 1}| = \lim_{n\to\infty} |x_n|$. Then we can easily verify that the function $|\{x_n\}^*_{n\geqq 1}|$ on L^* is a valuation of L^* and the original valuation $|\cdot|$ of L is naturally extended to this valuation of L^*.

Moreover, we have the following

Lemma 4. (1) *L is a dense subset of the metric space $(L^*, |\cdot|)$.*

(2) *$(L^*, |\cdot|)$ is a complete valued Lie algebra over $\mathfrak{k}$.*

Proof. (1) Let $x^* = \{x_n\}^*_{n\geqq 1} \in L^*$. For each integer $m \geqq 1$ let x^*_m denote the element of L represented by the constant sequence $\{x_m\}_{n\geqq 1}$. Since $\{x_n\}_{n\geqq 1} \in \mathrm{CS}(L)$ and

$$|x^*_m - x^*| = |\{x_m - x_n\}^*_{n\geqq 1}| = \lim_{n\to\infty} |x_m - x_n|,$$

we have $\lim_{m\to\infty} x_m^* = x^*$. Therefore L is a dense subset of the metric space $(L^*, |\cdot|)$.

(2) Let $\{x_n^*\}_{n\geqq 1}$ be a Cauchy sequence in $(L^*, |\cdot|)$. Given $\varepsilon > 0$, an integer $n_0 \geqq 1$ can be found with the property that $|x_m^* - x_n^*| < \varepsilon$ for all $m, n \geqq n_0$. Since L is a dense subset of the metric space $(L^*, |\cdot|)$, for each integer $n \geqq 1$ there exists an element y_n of L such that $|y_n - x_n^*| < \varepsilon$. Then for all $m, n \geqq n_0$ we have

$$|y_m - y^*| = |\{y_m - y_n\}_{n\geqq 1}^*| = \lim_{n\to\infty} |y_m - y_n| \leqq 3\varepsilon.$$

Thus we have $\lim_{m\to\infty} x_m^* = y^*$. Therefore $(L^*, |\cdot|)$ is a complete valued Lie algebra. □

By combining Lemmas 3 and 4, we can deduce the following theorem, which is the main result of this section.

Theorem 5. *Every valued Lie algebra has the unique completion up to isomorphism.*

3. ELEMENTARY PROPERTIES OF VALUED LIE ALGEBRAS

In this section we present some elementary properties of valued Lie algebras.

Let $(L, |\cdot|)$ be a valued Lie algebra over $\mathfrak{k}$. As in §2 we simply denote the completion of $(L, |\cdot|)$ by $(L^*, |\cdot|)$ from now on. For each subset X of L^* we denote by X^* the topological closure in the metric space $(L^*, |\cdot|)$. In particular, if X is a subspace of L^* then so is X^*. A subset X of L^* is said to be closed in $(L^*, |\cdot|)$, provided $X^* = X$.

Next, let $H \leq L^*$. Then $(H, |\cdot|)$ is a valued Lie algebra, where $|\cdot|$ means that the restriction of the valuation of L^* to H. Furthermore, we can clearly see that $(H, |\cdot|)$ is a complete valued Lie algebra if and only if H is closed in $(L^*, |\cdot|)$.

We begin with the following lemma, which is the Lie-theoretic analogue of the same result for normed linear spaces.

Lemma 6. *Let X be a subspace of L^* and assume that X is closed in $(L^*, |\cdot|)$. If $x_0 \in L^*\backslash X$, then $X + \langle x_0\rangle$ is closed in $(L^*, |\cdot|)$.*

Proof. Let $x_0 \in L^*\backslash X$ and let $\{y_n\}_{n\geqq 1}$ be a convergent sequence in $(L^*, |\cdot|)$ with limit $y^* \in L^*$ such that $y_n \in X + \langle x_0\rangle$ for all $n \geqq 1$. Then it suffices to prove that $y^* \in X + \langle x_0\rangle$. For each integer $n \geqq 1$ there exist $z_n \in X$ and $\alpha_n \in \mathfrak{k}$ such that $y_n = z_n - \alpha_n x_0$. Since X is closed in $(L^*, |\cdot|)$, we can find a real number $\varepsilon > 0$ such that $|x - x_0| \geqq \varepsilon$ for all $x \in X$. Moreover, there exists an integer $n_0 \geqq 1$ such that $|y_m - y_n| < \varepsilon$ for all $m, n \geqq n_0$. Now we suppose that there exist integers $m, n \geqq n_0$ such that $\alpha_m \neq \alpha_n$ and $m \neq n$. Since $\frac{1}{\alpha_m - \alpha_n}(z_m - z_n) \in X$, we have

$$\begin{aligned}|y_m - y_n| &= |(z_m - z_n) - (\alpha_m - \alpha_n)x_0| \\ &= \left|\frac{1}{\alpha_m - \alpha_n}(z_m - z_n) - x_0\right| \geqq \varepsilon.\end{aligned}$$

This is a contradiction. Hence $\alpha_m = \alpha_n$ for all $m, n \geqq n_0$. It follows that $\lim_{m,n\to\infty} |z_m - z_n| = \lim_{m,n\to\infty} |y_m - y_n| = 0$. Therefore we have $\{z_n\}_{n\geqq 1} \in \mathrm{CS}(L^*)$. Since $z_n \in X$ for all $n \geqq 1$ and X is closed in $(L^*, |\cdot|)$, we can find an element z^* of X such that $\lim_{n\to\infty} z_n = z^*$. Then for each $n \geqq n_0$ we have

$$\lim_{n\to\infty} |y_n - (z^* - \alpha_{n_0} x_0)| = \lim_{n\to\infty} |z_n - z^*| = 0.$$

Thus we obtain $y^* = \lim_{n\to\infty} y_n = z^* - \alpha_{n_0} x_0 \in X + \langle x_0 \rangle$. □

The following proposition is the first main result of this section.

Proposition 7. *Let X be a subspace of L^* and assume that X is closed in $(L^*, |\cdot|)$. If F is a finite-dimensional subspace of L^*, then $X + F$ is closed in $(L^*, |\cdot|)$. In particular, every finite-dimensional subspace of L^* is closed in $(L^*, |\cdot|)$.*

Proof. By using induction on $n = \dim F$, we can easily deduce from Lemma 6 that $X + F$ is closed in $(L^*, |\cdot|)$. Since $\{0\}$ is closed in $(L^*, |\cdot|)$, every finite-dimensional subspace of L^* is closed in $(L^*, |\cdot|)$. □

Lemma 8. (1) *If X and Y are non-empty subsets of L^* then $[X^*, Y^*] \subseteq [X, Y]^*$.*

(2) *Let $n < \omega$ and let Δ denote any one of the relations $\leq, \leq^n$ and $\lhd^n$. If $H \Delta L$ then $H^* \Delta L^*$.*

Proof. (1) Let $x^* \in X^*$ and $y^* \in Y^*$. Then there exists a convergent sequence $\{x_n\}_{n \geqq 1}$ (resp. $\{y_n\}_{n \geqq 1}$) in $(L^*, |\cdot|)$ with limit x^* (resp. y^*) such that $x_n \in X$ (resp. $y_n \in Y$) for all $n \geqq 1$. Then by Lemma 2 we have $[x^*, y^*] = \lim_{n\to\infty}[x_n, y_n] \in [X, Y]^*$.

(2) If Δ denotes $\leq$ then the result is immediately deduced from (1). Let $n < \omega$ and let Δ denote $\leq^n$ (resp. $\lhd^n$). Assume that $H \Delta L$. Then there exists a weak series (resp. a series) $(H_i)_{i \leqq n}$ from H to L. Using (1) we can show that $(H_i^*)_{i \leqq n}$ is a weak series (resp. a series) from H^* to L^*. It follows that $H^* \Delta L^*$. □

Let $H \leq L$. By Lemma 8 (2) we have $H^* \leq L^*$. Moreover, it is clear that $(H^*)^* = H^*$. Thus we obtain the following

Proposition 9. *If $H \leq L$ then $(H^*, |\cdot|)$ is the completion of the valued Lie algebra $(H, |\cdot|)$.*

Let F be a free Lie algebra over $\mathfrak{k}$ on a countably infinite set $\{t_1, t_2, \dots\}$ and Ω a set of words of F. Then the variety corresponding to Ω is denoted by $\mathfrak{V}_\Omega$ (cf. [1, p. 275]).

Finally we present the second main result of this section in the following

Theorem 10. *Let $n < \omega$ and let Δ denote any one of the relations $\leq, \leq^n$ and $\lhd^n$. Assume that $H \Delta L$.*

(1) *If $H \in \mathfrak{V}_\Omega$ then $H^* \in \mathfrak{V}_\Omega$ and $H^* \Delta L^*$.*

(2) *Let $\mathfrak{X}$ denote any one of the classes $\mathfrak{F}, \mathrm{E}\mathfrak{A}$ and $\mathfrak{N}$. If $H \in \mathfrak{X}$ then $H^* \in \mathfrak{X}$ and $H^* \Delta L^*$.*

Proof. It is immediately deduced from Lemma 8 that $H^* \Delta L^*$.

(1) Let $w \in \Omega$. We may suppose that the word w is in variables $t_1, \dots, t_r$, that is, $w = w(t_1, \dots, t_r)$. Assume that $H \in \mathfrak{V}_\Omega$ and let $x_i^* \in H^*$ $(1 \leqq i \leqq r)$. For each i there exists a convergent sequence $\{x_{i,n}\}_{n \geqq 1}$ in $(L^*, |\cdot|)$ with limit x_i^* such that $x_{i,n} \in H$ for all $n \geqq 1$. Then by Lemma 2 we have

$$w(x_1^*, \dots, x_r^*) = \lim_{n\to\infty} w(x_{1,n}, \dots, x_{r,n}) = 0.$$

It follows that $H^* \in \mathfrak{V}_\Omega$.

(2) Assume that $H \in \mathfrak{X}$. If $\mathfrak{X}$ denotes $\mathfrak{F}$ then by Proposition 7 we have $H^* = H \in \mathfrak{F}$. Let $\mathfrak{X}$ denote $\mathrm{E}\mathfrak{A}$ (resp. $\mathfrak{N}$). Then we can find an integer $n \geqq 1$ such that $H \in \mathfrak{A}^n$ (resp. $\mathfrak{N}_n$). Since $\mathfrak{A}^n$ (resp. $\mathfrak{N}_n$) is a variety corresponding to some single word of F, by (1) we have $H^* \in \mathfrak{A}^n$ (resp. $\mathfrak{N}_n$). Thus we obtain $H^* \in \mathfrak{X}$. □

4. VALUATIONS OF GENERALIZED WITT ALGEBRAS

In this section we confine our attention to valuations of generalized Witt algebras over a field $\mathfrak{k}$ of characteristic zero.

Let G be a non-trivial subgroup of the additive group $\mathfrak{k} \cap \mathbb{R}$ and let W_G denote a Lie algebra over $\mathfrak{k}$ with basis $\{w_g : g \in G\}$ and multiplication

$$[w_g, w_h] = (g-h)w_{g+h} \quad (g, h \in G).$$

These Lie algebras W_G are usually called generalized Witt algebras (cf. [1, p. 206]).

Let $0 \neq x \in W_G$. There exists a non-empty subset $J(x)$ of G such that $x = \sum_{g \in J(x)} \lambda_g w_g$ and $\lambda_g \in \mathfrak{k}^\times$ $(g \in J(x))$. Regard G as a linearly ordered set with respect to the linear ordering $<$ on $\mathbb{R}$. As in [1, p. 207] we define

$$\max(x) = \max\{g : g \in J(x)\} \quad \text{and} \quad \min(x) = \min\{g : g \in J(x)\}.$$

For convenience' sake, we also define

$$\max(0) = -\infty \quad \text{and} \quad \min(0) = +\infty.$$

It is not hard to verify that the function $\max(x)$ (resp. $\min(x)$) defined in W_G satisfies the following conditions (1) — (3) (resp. (1') — (3')), where $x, y \in W_G$ and $\alpha \in \mathfrak{k}^\times$:

(1) $\max(\alpha x) = \max(x)$
(resp. (1') $\min(\alpha x) = \min(x)$).
(2) $\max(x+y) \leqq \max\{\max(x), \max(y)\}$
(resp. (2') $\min(x+y) \geqq \min\{\min(x), \min(y)\}$).
(3) $\max([x,y]) \leqq \max(x) + \max(y)$
(resp. (3') $\min([x,y]) \geqq \min(x) + \min(y)$).

Hence we can define a non-trivial valuation $|\cdot|$ (resp. $|\cdot|'$) of W_G by

$$|x| = 2^{\max(x)} \quad (x \in W_G) \quad (\text{resp. } |x|' = (1/2)^{\min(x)} \quad (x \in W_G)).$$

Remark. Let c be a real number > 1. Then we can also define a non-trivial valuation $|\cdot|_1$ (resp. $|\cdot|'_1$) of W_G by

$$|x|_1 = c^{\max(x)} \quad (x \in W_G) \quad (\text{resp. } |x|'_1 = (1/c)^{\min(x)} \quad (x \in W_G)).$$

It is easy to see that the topology of $(W_G, |\cdot|_1)$ (resp. $(W_G, |\cdot|'_1)$) coincides with the topology of $(W_G, |\cdot|)$ (resp. $(W_G, |\cdot|')$).

Now we prove that the valued Lie algebra $(W_G, |\cdot|)$ is isomorphic to the valued Lie algebra $(W_G, |\cdot|')$. Let θ be a linear endomorphism of W_G defined by $\theta(w_g) = -w_{-g}$ $(g \in G)$. Then it is clear that θ is a Lie endomorphism of W_G. Since $\theta \circ \theta = \mathrm{id}$, θ is a Lie automorphism of W_G. Let $0 \neq x \in W_G$. Obviously we have $J(\theta(x)) = \{-g : g \in J(x)\}$. It follows that $\min(\theta(x)) = -\max(x)$. Since

$$|\theta(x)|' = (1/2)^{\min(\theta(x))} = 2^{\max(x)} = |x|,$$

θ is an isomorphism from $(W_G, |\cdot|)$ to $(W_G, |\cdot|')$. Consequently we may treat these two valued Lie algebras $(W_G, |\cdot|)$ and $(W_G, |\cdot|')$ without discrimination. So we consider the valuation $|\cdot|$ of W_G only.

Let $(W_G^*, |\cdot|)$ be the completion of the valued Lie algebra $(W_G, |\cdot|)$. We begin with the following

Proposition 11. *The valued Lie algebra* $(W_G, |\cdot|)$ *is not complete, that is,* $W_G \neq W_G^*$.

Proof. Assume that $(W_G, |\cdot|)$ is complete. Since G is a non-trivial subgroup of the additive group $\mathfrak{k} \cap \mathbb{R}$, there exists an element g of G such that $g > 0$ and $(\mathbb{Z} \cong)\mathbb{Z}g \leq G$. For each integer $n \geqq 1$ we put

$$x_n = w_{-ng} + \cdots + w_{-g} + w_0 \in W_G.$$

If $m > n \geqq 1$ then $\max(x_m - x_n) = -(n+1)g$. Hence $\{x_n\}_{n \geqq 1}$ is a Cauchy sequence in $(W_G, |\cdot|)$. Then there exists an element x of W_G such that $\lim_{n\to\infty} x_n = x$. Since $|x_n| = 1$ for all $n \geqq 1$, we have $x \neq 0$. Let $x = \sum_{h \in J(x)} \lambda_h w_h$ and $\lambda_h \in \mathfrak{k}^\times$ $(h \in J(x))$. Then we can find an integer $N \geqq 1$ such that $-ng < \min(x)$ for all $n \geqq N$. Therefore we have $\max(x_n - x) \geqq -Ng$. It follows that $|x_n - x| \geqq 2^{-Ng}$ for all $n \geqq N$. Thus we obtain $\lim_{n\to\infty} |x_n - x| \neq 0$. This is a contradiction. □

Furthermore, we restrict our attention to the case $G = \mathbb{Z}$. Then we have the following theorem which is the main result of this section.

Theorem 12. *If* $0 \neq H$ wsi $\mathrm{W}_{\mathbb{Z}}$ *then* H *is a dense subset of the metric space* $(W_{\mathbb{Z}}^*, |\cdot|)$.

Proof. Assume that $0 \neq H \leq^n W_{\mathbb{Z}}$ for some integer $n \geqq 1$. Then we can find a non-zero element $x = \sum_{g<N} \lambda_g w_g + w_N \in H$, where $N = \max(x)$ and $\lambda_g \in \mathfrak{k}$ $(g < N)$. Put $-m = \min\{(n+1)N, 0\} - 2$. Then $-m \leqq -2$. Let k be an integer $\leqq -m$ and put $l = k - nN$. Then it is easy to see that $k < \min\{(n+1)N, 0\}$. Therefore $l + (j-1)N \neq 0$ $(j = 0, 1, \ldots, n+1)$. Thus we have

$$[w_l,\ _n x] = \sum_{g<k} \lambda'_g w_g + (l-N)l \cdots \{l - (n-2)N\} w_k \in H,$$

where $\lambda'_g \in \mathfrak{k}$ $(g < k)$ and $(l-N)l \cdots \{l-(n-2)N\} \neq 0$. Hence there exists a non-zero element $x_k = \sum_{g<k} \lambda_{k,g} w_g + w_k \in H$ such that $\lambda_{k,g} \in \mathfrak{k}$ $(g < k)$.

Now we recursively define the terms of a sequence $\{y_i\}_{i \geqq 1}$ such that $y_i \in H$ and $\max(y_i - w_{-m}) \leqq -(m+i)$ for each $i \geqq 1$. First define $y_1 = x_{-m} \in H$. Hence $y_1 - w_{-m} = \sum_{g<-m} \lambda_{-m,g} w_g$. It follows that $\max(y_1 - w_{-m}) \leqq -(m+1)$. Let $i \geqq 1$ and suppose that the terms y_j $(j = 1, \ldots, i)$ have been defined. There exist $\mu_g \in \mathfrak{k}$ $(g \leqq -(m+i))$ such that $y_i - w_{-m} = \sum_{g<-(m+i)} \mu_g w_g + \mu_{-(m+i)} w_{-(m+i)}$. Next define $y_{i+1} = y_i - \mu_{-(m+i)} x_{-(m+i)} \in H$. Then

$$\begin{aligned} y_{i+1} - w_{-m} &= y_i - w_{-m} - \mu_{-(m+i)} \left\{ \sum_{g<-(m+i)} \lambda_{-(m+i),g} w_g + w_{-(m+i)} \right\} \\ &= \sum_{g<-(m+i)} (\mu_g - \mu_{-(m+i)} \lambda_{-(m+i),g}) w_g \end{aligned}$$

and hence $\max(y_{i+1} - w_{-m}) \leqq -(m+i+1)$. Thus we can construct such a sequence $\{y_i\}_{i \geqq 1}$. Then

$$|y_i - w_{-m}| \leqq 2^{-(m+i)} \quad (i = 1, 2, \ldots).$$

Therefore we have $w_{-m} = \lim_{i\to\infty} y_i \in H^*$. It follows from Lemma 8 (2) that $H^* \leq^n W_{\mathbb{Z}}^*$. Hence $[z,\ _n w_{-m}] \in H^*$ for all $z \in W_{\mathbb{Z}}$.

Let p be an integer $\neq -im$ $(i = 2, 3, \ldots, n+1)$. Put $q = p + nm$. Then

$$[w_q,\ _n w_{-m}] = (q+m)q(q-m) \cdots \{q - (n-2)m\} w_p \in H^*$$

and $(q+m)q(q-m)\cdots\{q-(n-2)m\} \neq 0$. Hence we have $w_p \in H^*$. Since $-m \leqq -2$, we have $w_p \in H^*$ $(p = -3, -2, \dots)$. It follows that $W_{\mathbb{Z}} = \langle w_{-2}, w_{-1}, w_1, w_2 \rangle \leq H^*$. Thus we obtain $H^* = W_{\mathbb{Z}}^*$. □

Corollary 13. *If* H wsi $\mathrm{W}_{\mathbb{Z}}$ *and* $H \in \mathfrak{F} \cup \mathrm{E}\mathfrak{A}$ *then* $H = 0$.

Proof. [2, Corollary (c)] asserts that every soluble subalgebra of $W_{\mathbb{Z}}$ is of dimension $\leqq 2$. Hence we may suppose that H wsi $\mathrm{W}_{\mathbb{Z}}$ and $H \in \mathfrak{F}$. Owing to Proposition 7, we have $H^* = H \in \mathfrak{F}$. Assume that $H \neq 0$. Using Theorem 12, we have $W_{\mathbb{Z}}^* = H^* \in \mathfrak{F}$, a contradiction. □

Finally we construct an example showing that if we replace 'H wsi $\mathrm{W}_{\mathbb{Z}}$' by '$H \leq^{\omega} W_{\mathbb{Z}}$' in the statement of Theorem 12 then it becomes a failure.

Example 14. Put $H = \sum_{g \geqq 1} \langle w_g \rangle$. Then H is an infinite-dimensional subalgebra of $W_{\mathbb{Z}}$. Moreover, put $H_0 = H, H_n = \sum_{g \geqq 1-n} \langle w_g \rangle$ $(1 \leqq n < \omega)$ and $H_\omega = W_{\mathbb{Z}}$. Then it is clear that $(H_n)_{n \leqq \omega}$ is a weakly ascending series from H to L, that is, $H \leq^{\omega} W_{\mathbb{Z}}$. On the other hand, we can easily show that $|x| = 2^{\max(x)} \geqq 2$ for all $x \in H \backslash \{0\}$. Thus we obtain $H^* = H \neq W_{\mathbb{Z}}^*$.

5. GENERALIZED WITT ALGEBRAS OF TYPE $\pm\infty$

In this section we construct new Lie algebras $W_{-\infty}$ and $W_{+\infty}$ which contain the generalized Witt algebra $W_{\mathbb{Z}}$ over a field $\mathfrak{k}$ of characteristic zero.

Let $\{w_r : r \in \mathbb{Z}\}$ be a basis for generalized Witt algebra $W_{\mathbb{Z}}$ and $\langle w_r \rangle = \mathfrak{k} w_r$ 1-dimensional vector spaces $(r \in \mathbb{Z})$. We consider the Cartesian sum $\mathrm{Cr}_{r \in \mathbb{Z}} \langle w_r \rangle$ of $\langle w_r \rangle$ $(r \in \mathbb{Z})$. Then we make $\mathrm{Cr}_{r \in \mathbb{Z}} \langle w_r \rangle$ into a vector space by defining an addition and a scalar multiplication pointwise.

Now we set a subspace $W_{-\infty}$ (resp. $W_{+\infty}$) of $\mathrm{Cr}_{r \in \mathbb{Z}} \langle w_r \rangle$ as follows:

$$W_{-\infty} = \left\{ \sum_{r \leqq n} \lambda_r w_r : n \in \mathbb{Z},\ \lambda_r \in \mathfrak{k}\ (r \leqq n) \right\}$$

$$\left(\text{resp.} \quad W_{+\infty} = \left\{ \sum_{r \geqq n} \lambda_r w_r : n \in \mathbb{Z},\ \lambda_r \in \mathfrak{k}\ (r \geqq n) \right\} \right).$$

Moreover we can define a bracket product in $W_{-\infty}$ (resp. $W_{+\infty}$) as follows: For $x = \sum_{r \leqq n} \lambda_r w_r,\ y = \sum_{s \leqq m} \mu_s w_s \in W_{-\infty}$ (resp. $x = \sum_{r \geqq n} \lambda_r w_r,\ y = \sum_{s \geqq m} \mu_s w_s \in W_{+\infty}$),

$$[x, y] = \sum_{t \leqq n+m} \left(\sum_{r+s=t} \lambda_r \mu_s (r - s) \right) w_t$$

$$\left(\text{resp.} \quad [x, y] = \sum_{t \geqq n+m} \left(\sum_{r+s=t} \lambda_r \mu_s (r - s) \right) w_t \right).$$

It is not hard to formulate that $W_{-\infty}$ and $W_{+\infty}$ are Lie algebras containing $W_{\mathbb{Z}}$ as a subalgebra. We call $W_{-\infty}$ (resp. $W_{+\infty}$) the generalized Witt algebra of type $-\infty$ (resp. $+\infty$).

Let $0 \neq x \in W_{-\infty}$ (resp. $W_{+\infty}$) and $x = \sum_{r \leqq n} \lambda_r w_r$ (resp. $\sum_{r \geqq n} \lambda_r w_r$) $(\lambda_n \neq 0)$. Then, as in §4, we define

$$\max(x) = n \text{ (resp. } \min(x) = n) \quad \text{and} \quad \max(0) = -\infty \text{ (resp. } \min(0) = +\infty).$$

It is easy to show that the function $\max(x)$ (resp. $\min(x)$) satisfies the conditions (1) — (3) (resp. (1') — (3')) in §4. Hence we can also define a non-trivial valuation $|\cdot|$ (resp. $|\cdot|'$) of $W_{-\infty}$ (resp. $W_{+\infty}$) by

$$|x| = 2^{\max(x)} \quad (x \in W_{-\infty}) \quad (\text{resp. } \ |x|' = (1/2)^{\min(x)} \quad (x \in W_{+\infty})).$$

Now we prove that the valued Lie algebra $(W_{-\infty}, |\cdot|)$ is isomorphic to the valued Lie algebra $(W_{+\infty}, |\cdot|')$. Let θ be a linear mapping from $W_{-\infty}$ to $W_{+\infty}$ defined by

$$\theta(\sum_{r \leqq n} \lambda_r w_r) = \sum_{r \leqq n} (-\lambda_r) w_{-r}.$$

Similarly, let θ' be a linear mapping from $W_{+\infty}$ to $W_{-\infty}$ defined by

$$\theta'(\sum_{r \geqq n} \lambda_r w_r) = \sum_{r \geqq n} (-\lambda_r) w_{-r}.$$

Then it is clear that θ (resp. θ') is a Lie homomorphism from $W_{-\infty}$ to $W_{+\infty}$ (resp. from $W_{+\infty}$ to $W_{-\infty}$). Since $\theta \circ \theta' = \mathrm{id}_{W_{+\infty}}$ and $\theta' \circ \theta = \mathrm{id}_{W_{-\infty}}$, θ becomes a Lie isomorphism. Let $0 \neq x \in W_{-\infty}$ and $x = \sum_{r \leqq n} \lambda_r w_r$ $(\lambda_n \neq 0)$. As $\theta(x) = \sum_{r \leqq n} (-\lambda_r) w_{-r}$, we have

$$|\theta(x)|' = (1/2)^{\min(\theta(x))} = (1/2)^{-n} = 2^n = 2^{\max(x)} = |x|.$$

Therefore θ is an isomorphism from $(W_{-\infty}, |\cdot|)$ to $(W_{+\infty}, |\cdot|')$. Furthermore, we can easily see that $\theta(W_{\mathbb{Z}}) = W_{\mathbb{Z}}$ and $\theta'(W_{\mathbb{Z}}) = W_{\mathbb{Z}}$.

Next we give an important property of the valued Lie algebra $(W_{-\infty}, |\cdot|)$, which is a main result in this section.

Theorem 15. $(W_{-\infty}, |\cdot|)$ *is complete and* $W_{\mathbb{Z}}$ *is a dense subset of the metric space* $(W_{-\infty}, |\cdot|)$.

Proof. Let $\{x_n\}_{n \geqq 1}$ be a Cauchy sequence in $(W_{-\infty}, |\cdot|)$. If there is an infinite number of positive integers n such that $x_n = 0$, then we can find a subsequence $\{x_{n(k)}\}_{k \geqq 1}$ of $\{x_n\}_{n \geqq 1}$ such that $x_{n(k)} = 0$ for all $k \geqq 1$. Hence we get $\lim_{n \to \infty} x_n = 0$. Otherwise we may assume that there is a finite number of n with $x_n = 0$. Thus we suppose that $x_n \neq 0$ for all $n \geqq 1$. Set $\max(x_n) = l(n)$ $(n \geqq 1)$ and let the expression of x_n be

$$x_n = \sum_{i \leqq l(n)} \lambda_{n,i} w_i \quad (\lambda_{n,i} \in \mathfrak{k} \ \ (i \leqq l(n)), \ \lambda_{n,l(n)} \neq 0).$$

For any $k \geqq 1$ there exists an ascending natural number $N(k)$ such that $|x_m - x_n| < 2^{-k}$ for all $m, n \geqq N(k)$. That is to say, $\max(x_m - x_n) < -k$ for all $m, n \geqq N(k)$. Since

$$x_m - x_n = \sum_{i \leqq \max\{l(m), \ l(n)\}} (\lambda_{m,i} - \lambda_{n,i}) w_i,$$

the condition $\max(x_m - x_n) < -k$ for all $m, n \geqq N(k)$ means either

(i) $l(m) < -k$ and $l(n) < -k$ for all $m, n \geqq N(k)$, or

(ii) $l(m) = l(n) \geqq -k$ and $\lambda_{m,i} = \lambda_{n,i}$ $(i = -k, -(k-1), \ldots, l(m) = l(n))$ for all $m, n \geqq N(k)$.

If the condition (i) holds for all $k \in \mathbb{N}$, then for all $n \geqq N(k)$

$$|x_n| = 2^{\max(x_n)} = 2^{l(n)} < 2^{-k} \to 0 \quad (k \to \infty).$$

Hence we obtain $\lim_{n\to\infty} x_n = 0$. Thus we suppose that the condition (ii) holds for some $k \in \mathbb{N}$. Since it is enough to show that $\{x_n\}_{n \geqq N(k)}$ is convergent, we replace $\{x_n\}_{n \geqq N(k)}$ with $\{x_n\}_{n \geqq 1}$. Therefore we may suppose that

$$l(m) = l(n) \quad \text{for all} \quad m, n \geqq 1.$$

Putting $l = l(m)$ we get

$$x_m - x_n = \sum_{i \leqq l} (\lambda_{m,i} - \lambda_{n,i}) w_i \quad \text{for all } m, n \geqq 1.$$

By using the similar argument above, we obtain that for any $k \geqq 1$ there exists an ascending natural number $N(k)$ such that $\max(x_m - x_n) < -k$ for all $m, n \geqq N(k)$, that is, either

(i) $l < -k$, or
(ii) $l \geqq -k$ and $\lambda_{m,i} = \lambda_{n,i}$ $(i = -k, -(k-1), \ldots, l)$ for all $m, n \geqq N(k)$.

Now we consider two cases:

Case 1. Let $l < 0$. Then for each $k \in \mathbb{N}$ with $l \geqq -k$ it follows from (ii) that

$$\lambda_{N(k),-k} = \lambda_{N(k)+1,-k} = \cdots = \lambda_{n,-k}, \text{ say } \lambda_{-k}, \quad \text{for all} \quad n \geqq N(k).$$

Here we set $x^* = \sum_{i \leqq l} \lambda_i w_i \in W_{-\infty}$.

Case 2. Let $l \geqq 0$. Then for each $k \in \mathbb{N}$ it follows from (ii) that

$$\lambda_{N(k),-k} = \lambda_{N(k)+1,-k} = \cdots = \lambda_{n,-k}, \text{ say } \lambda_{-k}, \quad \text{for all} \quad n \geqq N(k).$$

Since $l \geqq -1$ for $k = 1$ in the condition (ii), we have

$$\begin{aligned}
\lambda_{N(1),0} &= \lambda_{N(1)+1,0} = \cdots = \lambda_{n,0}, \text{ say } \lambda_0, \quad \text{for all} \quad n \geqq N(1), \\
\lambda_{N(1),1} &= \lambda_{N(1)+1,1} = \cdots = \lambda_{n,1}, \text{ say } \lambda_1, \quad \text{for all} \quad n \geqq N(1), \\
&\cdots\cdots \\
\lambda_{N(1),l} &= \lambda_{N(1)+1,l} = \cdots = \lambda_{n,l}, \text{ say } \lambda_l, \quad \text{for all} \quad n \geqq N(1).
\end{aligned}$$

Here we also set $x^* = \sum_{i \leqq l} \lambda_i w_i \in W_{-\infty}$.

Let $k \in \mathbb{N}$. Then for any $n \geqq N(k)$ we have

$$x_n - x^* = \sum_{i \leqq l} (\lambda_{n,i} - \lambda_i) w_i = \sum_{i \leqq -(k+1)} (\lambda_{n,i} - \lambda_i) w_i.$$

Hence we obtain $\max(x_n - x^*) \leqq -(k+1) < -k$. Thus

$$|x_n - x^*| = 2^{\max(x_n - x^*)} < 2^{-k} \to 0 \ (k \to \infty).$$

That is to say, $\lim_{n\to\infty} x_n = x^* \in W_{-\infty}$. Therefore we conclude that $(W_{-\infty}, |\cdot|)$ is complete.

Finally we must claim that $W_{\mathbb{Z}}$ is a dense subset of $(W_{-\infty}, |\cdot|)$. Let $x = \sum_{r \leqq n} \lambda_r w_r$ $(\lambda_n \neq 0)$ be any non-zero element of $W_{-\infty}$. For any integer $m \geqq 1$ we define

$$x_m = \sum_{i=0}^{m} \lambda_{n-i} w_{n-i} \in W_{\mathbb{Z}}.$$

Given any $\varepsilon > 0$, there is an integer $N > 0$ with $2^{-N} < \varepsilon$. Then for all $m > N + |n|$ we have

$$|x_m - x| = \left| \sum_{r \leqq n-m-1} (-\lambda_r) w_r \right| < 2^{-(m-n)} < 2^{-N} < \varepsilon.$$

Therefore it follows that $\lim_{m \to \infty} x_m = x$. □

Accordingly we get the following corollary to this theorem.

Corollary 16. $(W_{-\infty}, |\cdot|)$ *is the completion of* $(W_{\mathbb{Z}}, |\cdot|)$.

Remark. Since $(W_{-\infty}, |\cdot|)$ is isomorphic to $(W_{+\infty}, |\cdot|')$, $(W_{+\infty}, |\cdot|')$ is also the completion of $(W_{\mathbb{Z}}, |\cdot|')$.

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